The gravitational self-force

[E. Poisson, *Living Reviews in Relativity* **7**, 6 (2004); gr-qc/0306052]

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1. Capra scientific mandate

To formulate the equations of motion of a small body of mass m in a specified background spacetime, beyond the test-mass approximation.

This first step was solved back in 1997. The equations of motion are now known as the MiSaTaQuWa equations. I will sketch a derivation of these equations.

To concretely describe this motion for situations of astrophysical interest (generic orbits of a Kerr black hole).

Much recent progress on this front. I will describe some of the issues involved.

To properly incorporate this information into a wave-generation formalism.

The holy grail: still elusive. I will present a tentative outline of future work.

The work reviewed in this talk was shaped by a series of seven "Capra" meetings, from 1998 to 2004:

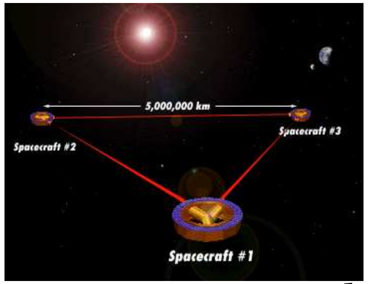
Capra Ranch, Dublin, Pasadena, Potsdam, State College PA, Kyoto, and Brownsville.

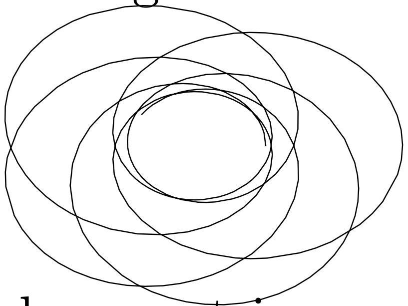
Members of the Capra posse include:

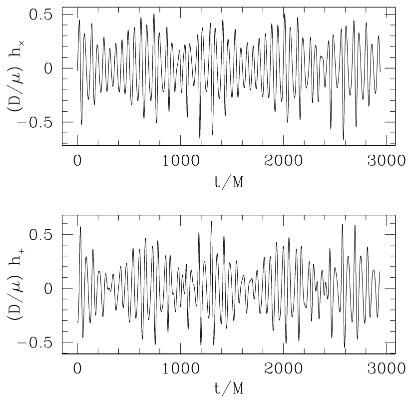
Paul Anderson, Warren Anderson, Leor Barack, Patrick Brady, Lior Burko, Manuella Campanelli, Steve Detweiler, Eanna Flanagan, Costas Glempedakis, Abraham Harte, Wataru Hikida, Bei Lok Hu, Scott Hughes, Sanjay Jhingan, Dong-Hoon Kim, Carlos Lousto, Eirini Messaritaki, Yasushi Mino, Hiroyuki Nakano, Amos Ori, Eric Poisson, Ted Quinn, Eran Rosenthal, Norichika Sago, Misao Sasaki, Takahiro Tanaka, Bob Wald, Bernard Whiting, and Alan Wiseman.

2. Introduction

Solar-mass compact objects moving around massive black holes are one of the promising sources of gravitational waves for LISA.







These systems involve highly eccentric, nonequatorial, and relativistic orbits around rapidly rotating black holes.

During a year's worth of observation, these sources will produce in excess of 10^5 wave cycles that must be modeled accurately for information extraction.

Modeling involves formulating equations of motion for the small body, in a **small-mass-ratio approximation** that goes beyond the test-mass limit.

Corrections to the equations of motion must incorporate

- conservative effects, such as periastron advance
  (which can lead to a large cumulative dephasing);
- dissipative effects, such as loss of orbital energy and angular momentum to gravitational radiation
  (which produces a frequency sweep in the LISA band).

Modeling also involves improving the wave-generation calculations to consistently incorporate

- the corrected equations of motion;
- the effect of the small body's gravitational field on wave generation and propagation.

3. Motion of a black hole

A body of mass m moves in an arbitrary (but empty) spacetime whose radius of curvature (in a neighbourhood of the body) is $\mathcal{R}$; we assume that $m/\mathcal{R} \ll 1$.

When viewed on a large scale $\mathcal{R}$, the body appears to move on a world line γ described by parametric relations $z^\mu(\tau)$; we wish to determine this world line.

A compelling way to proceed is to assume that the body is a **nonrotating black hole** [Mino, Sasaki, and Tanaka (1997)].

The metric of the black hole perturbed by the tidal gravitational field of the external universe is matched to the metric of the background spacetime perturbed by the moving black hole.

Demanding that this metric be a solution to the vacuum field equation determines the motion of the black hole.

The method of matched asymptotic expansions relies on the existence of

- an **internal zone** in which $r/\mathcal{R} \ll 1$, where r is the distance to the black hole (here gravity is dominated by the black hole);
- an **external zone** in which $m/r \ll 1$ (here gravity is dominated by the external universe);
- a **buffer zone** in which $r/\mathcal{R}$ and m/r are both small (here the black hole and the external universe have comparable gravities).

The metrics are matched in the buffer zone, where $m \ll r \ll \mathcal{R}$.

External zone:

The metric of the unperturbed external universe is written in (external) retarded coordinates (u, x^a) as an expansion in powers of $r \equiv$ (distance from world line γ, on which $u \equiv \tau$), eg,

$$g_{uu} = -\big(1 + 2r a_a \Omega^a + r^2 \mathcal{E}_{ab} \Omega^a \Omega^b\big) + O(r^3/\mathcal{R}^3)$$

where $\Omega^a = x^a/r$, $a_a(u)$ is the acceleration of the world line, and $\mathcal{E}_{ab}(u)$ are components of the Weyl tensor evaluated on γ.

The perturbation $h_{\alpha\beta}$ produced by the black hole is determined by the equations

$$\Box \psi^{\alpha\beta} + 2R^{\alpha\ \ \beta}_{\ \ \gamma\ \ \delta} \psi^{\gamma\delta} = -16\pi T^{\alpha\beta}, \qquad \psi^{\alpha\beta}{}_{;\beta} = 0$$

and

$$h_{\alpha\beta} = \psi_{\alpha\beta} - \frac{1}{2}\psi g_{\alpha\beta}$$

where $T^{\alpha\beta}$ is the stress-energy tensor of a point particle of mass m.

These equations are solved by means of a retarded Green's function

$$G^{\alpha\beta}{}_{\gamma'\delta'}(x, x') = U^{\alpha\beta}{}_{\gamma'\delta'}(x, x')\delta(\sigma) + V^{\alpha\beta}{}_{\gamma'\delta'}(x, x')\theta(-\sigma)$$

where σ is half the squared geodesic distance between x and x'.

The result is

$$\psi^{\alpha\beta}(x) = \frac{4m}{r} U^{\alpha\beta}{}_{\mu\nu}(x, z) u^\mu u^\nu + \psi^{\alpha\beta}_{\text{tail}}(x)$$

where $z^\mu(u)$ is the retarded point on the world line, $u^\mu = dz^\mu/d\tau$ is the four-velocity, and

$$\psi^{\alpha\beta}_{\text{tail}}(x) = 4m \int_{-\infty}^{u} V^{\alpha\beta}{}_{\mu'\nu'}(x, z') u^{\mu'} u^{\nu'}\, d\tau'$$

is the "tail" term.

All this gives, eg,

$$\begin{aligned}
\mathsf{g}_{uu} &= -1 - r^2\mathcal{E}_{ab}\Omega^a\Omega^b + O(r^3/\mathcal{R}^3) \\
&\quad + \frac{2m}{r} + h^{\text{tail}}_{00} + r\big(2m\mathcal{E}_{ab}\Omega^a\Omega^b - 2a_a\Omega^a + h^{\text{tail}}_{000} + h^{\text{tail}}_{00a}\Omega^a\big) \\
&\quad + O(mr^2/\mathcal{R}^3)
\end{aligned}$$

for the perturbed metric $\mathsf{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$.

The fields $h^{\text{tail}}_{\alpha\beta}$ are obtained from $\psi^{\text{tail}}_{\alpha\beta}$ by trace reversal, and $h^{\text{tail}}_{\alpha\beta\gamma} = \nabla_\gamma h^{\text{tail}}_{\alpha\beta}$.

Internal zone:

The metric of an unperturbed Schwarzschild black hole is written in (internal) retarded coordinates $(\bar{u}, \bar{x}^a)$ as, eg,

$$g_{\bar{u}\bar{u}} = -f = -\left(1 - \frac{2m}{\bar{r}}\right)$$

When the black hole is perturbed by a tidal gravitational field $\mathcal{E}_{ab}(\bar{u})$, its metric becomes, eg,

$$\mathsf{g}_{\bar{u}\bar{u}} = -f - \bar{r}^2 f^2 \mathcal{E}_{ab} \bar{\Omega}^a \bar{\Omega}^b + O(\bar{r}^3/\mathcal{R}^3)$$

Matching:

To match the metrics, the internal coordinates $(\bar{u}, \bar{x}^a)$ must first be related to the external coordinates (u, x^a), eg,

$$\begin{aligned} \bar{u} &= u - 2m \ln r - \frac{1}{2}\int^u h^{\mathrm{tail}}_{00}\, du - \frac{1}{2} r \Big[h^{\mathrm{tail}}_{00} + 2h^{\mathrm{tail}}_{0a}\Omega^a + h^{\mathrm{tail}}_{ab}\Omega^a\Omega^b \Big] \\ &\quad - \frac{1}{4} r^2 \Big[h^{\mathrm{tail}}_{000} + \big(h^{\mathrm{tail}}_{00a} + 2h^{\mathrm{tail}}_{0a0} \big)\Omega^a + \big(h^{\mathrm{tail}}_{ab0} + 2h^{\mathrm{tail}}_{0ab} \big)\Omega^a\Omega^b \\ &\quad + h^{\mathrm{tail}}_{abc}\Omega^a\Omega^b\Omega^c \Big] + O(mr^3/\mathcal{R}^3) \end{aligned}$$

After the transformation, the $\bar{u}$-$\bar{u}$ component of the external-zone metric becomes

$$\begin{aligned} \mathsf{g}_{\bar{u}\bar{u}} &= -1 - \bar{r}^2 \mathcal{E}_{ab}\bar{\Omega}^a\bar{\Omega}^b + O(\bar{r}^3/\mathcal{R}^3) \\ &\quad + \frac{2m}{\bar{r}} + \bar{r}\left[4m\mathcal{E}_{ab}\bar{\Omega}^a\bar{\Omega}^b - 2\left(a_a - \frac{1}{2} h^{\mathrm{tail}}_{00a} + h^{\mathrm{tail}}_{0a0} \right)\bar{\Omega}^a \right] \\ &\quad + O(m\bar{r}^2/\mathcal{R}^3) \end{aligned}$$

Matching with the internal-zone metric yields

$$a_a = \frac{1}{2} h^{\rm tail}_{00a} - h^{\rm tail}_{0a0}$$

for the components of the acceleration vector.

The tensorial form of the equations of motion is

$$\frac{Du^\mu}{d\tau} = -\frac{1}{2}\bigl(g^{\mu\nu} + u^\mu u^\nu\bigr)\bigl(2h^{\rm tail}_{\nu\lambda\rho} - h^{\rm tail}_{\lambda\rho\nu}\bigr)u^\lambda u^\rho$$

where

$$h^{\rm tail}_{\mu\nu\lambda} = 4m \int_{-\infty}^{\tau-\epsilon} \nabla_\lambda \biggl(G_{\mu\nu\mu'\nu'} - \frac{1}{2} g_{\mu\nu} G^{\rho}{}_{\rho\mu'\nu'} \biggr)\bigl(z(\tau), z(\tau')\bigr) u^{\mu'} u^{\nu'}\, d\tau'$$

These are the MiSaTaQuWa equations of motion for a nonrotating black hole of mass m [Mino, Sasaki, and Tanaka (1997); Quinn and Wald (1997)].

4. Motion of a point mass

The MiSaTaQuWa equations of motion are insensitive to the hole's internal structure; they should apply to any nonrotating body.

Can they be derived on the basis of a point particle? [Mino, Sasaki, and Tanaka (1997); Quinn and Wald (1997)]

The gravitational perturbation $h_{\alpha\beta}$ produced by a point particle is obtained from

$$\psi^{\alpha\beta}(x) = \frac{4m}{r} U^{\alpha\beta}{}_{\mu\nu}(x,z) u^\mu u^\nu + \psi^{\alpha\beta}_{\rm tail}(x)$$

by trace reversal.

This was previously assumed to hold in the external zone ($r \gg m$) only; this now holds all the way down to the world line, $r \to 0$.

The particle is **assumed** to move on a geodesic of the perturbed metric $\mathsf{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$.

This leads to the equations of motion

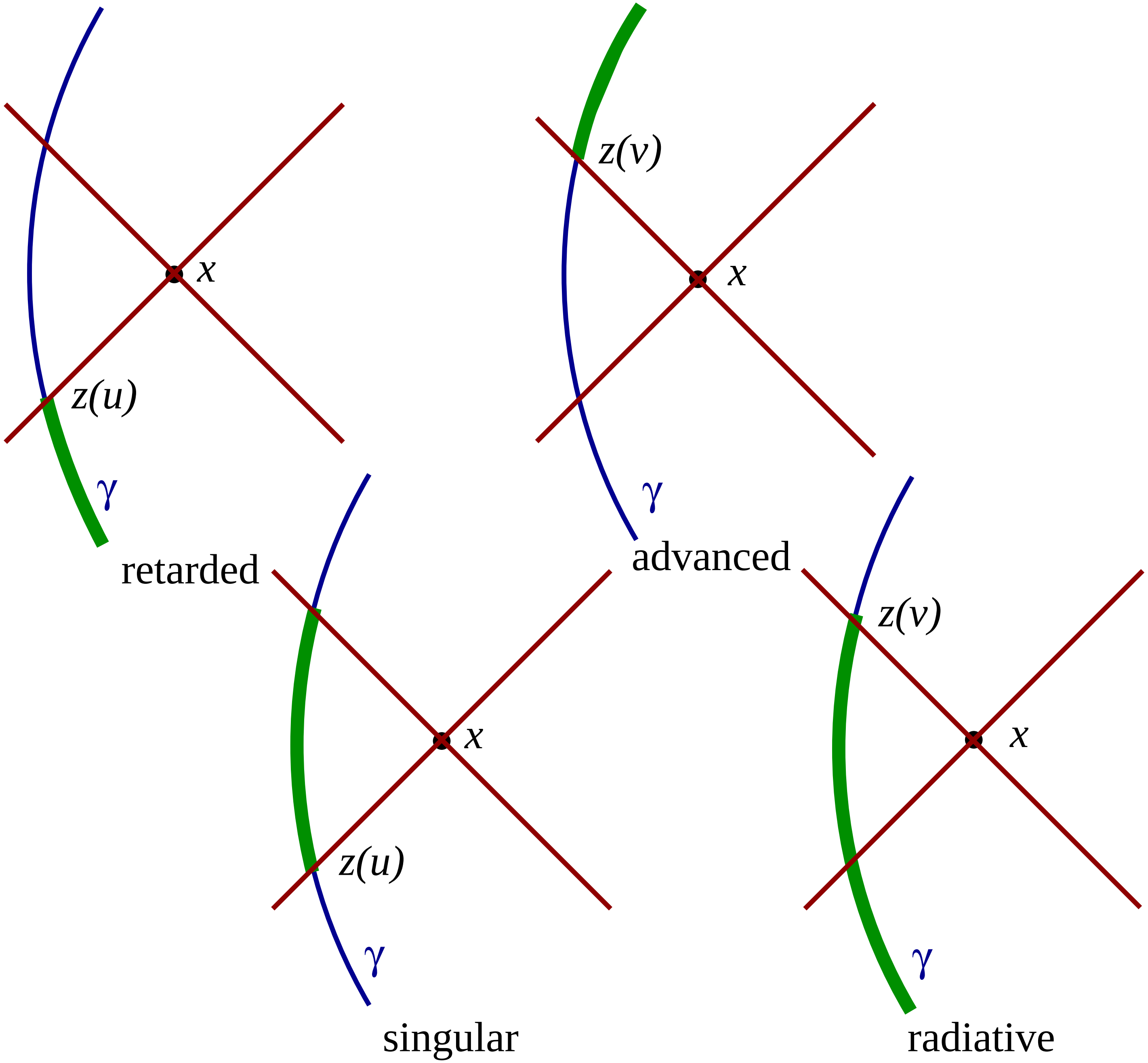
$$\frac{Du^\mu}{d\tau} = -\frac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(2h_{\nu\lambda;\rho} - h_{\lambda\rho;\nu}\right)u^\lambda u^\rho$$

But since $h_{\alpha\beta}$ is singular on the world line, this must be regularized before meaning can be assigned to this equation.

It is possible to uniquely decompose the retarded field $\psi_{\alpha\beta}$ into a **"singular" part** $\psi^{\mathrm{S}}_{\alpha\beta}$ and a **"radiative" part** $\psi^{\mathrm{R}}_{\alpha\beta}$ such that [Detweiler and Whiting (2003)]

$$\psi = \psi^{\mathrm{S}} + \psi^{\mathrm{R}}, \qquad (\Box + 2R)\psi^{\mathrm{S}} = -16\pi T, \qquad (\Box + 2R)\psi^{\mathrm{R}} = 0$$

and such that $\psi^{\mathrm{S}}_{\alpha\beta}$ is just as singular as $\psi_{\alpha\beta}$ but **exerts no force on the particle**.


x
z(u)
γ
retarded
z(v)
x
γ
advanced
x
z(u)
γ
singular
z(v)
x
γ
radiative

The equations of motion of a point mass are therefore

$$\frac{Du^\mu}{d\tau} = -\frac{1}{2}\big(g^{\mu\nu} + u^\mu u^\nu\big)\big(2h^{\rm R}_{\nu\lambda;\rho} - h^{\rm R}_{\lambda\rho;\nu}\big)u^\lambda u^\rho$$

On the world line

$$h^{\rm R}_{\mu\nu;\lambda} = -4m\Big(u_{(\mu}R_{\nu)\rho\lambda\xi} + R_{\mu\rho\nu\xi}u_\lambda\Big)u^\rho u^\xi + h^{\rm tail}_{\mu\nu\lambda}$$

and this result agrees with the black-hole equations of motion (the Riemann tensor terms cancel out).

The MiSaTaQuWa equations of motion can therefore be derived on the basis of a point mass, with a suitable regularization procedure.

They are formally equivalent to the statement that **the motion is geodesic in the metric $g_{\alpha\beta} + h^{\rm R}_{\alpha\beta}$**, which is a smooth solution to the vacuum field equations.

5. Newtonian self-force

To help illustrate the preceding decomposition of the gravitational perturbation into singular “S” and radiative “R” pieces, consider an analogous problem in Newtonian theory.

Here we have a large mass M at a distance $\boldsymbol{\rho}$ from the centre of mass, and a small mass m at a distance $\boldsymbol{R}$; we have $m \ll M$ and $m\boldsymbol{R} + M\boldsymbol{\rho} = \mathbf{0}$.

A test-mass description has m moving in a potential $\Phi_0 = -M/r$.

An improved description involves a potential $\Phi = \Phi_0 + \delta\Phi$, where

$$\delta\Phi = \underbrace{-\frac{M}{|\boldsymbol{x} - \boldsymbol{\rho}|} + \frac{M}{r}}_{\Phi_{\mathrm{R}}} \underbrace{- \frac{m}{|\boldsymbol{x} - \boldsymbol{R}|}}_{\Phi_{\mathrm{S}}}$$

Here the perturbation $\delta\Phi$ is decomposed into singular “S” and regular “R” pieces.

The singular potential is isotropic about m and it therefore exerts no force.

The regular potential is

$$\Phi_{\mathrm{R}} = m\frac{\boldsymbol{R}\cdot\boldsymbol{x}}{r^3} + O(m^2)$$

and it does exert a force on m.

For example, the Newtonian self-force gives rise to a finite-mass correction to the particle's angular velocity.

The perturbation $\delta\Phi$ and its singular part Φ_{S} contain an infinite number of multipole moments.

The regular potential Φ_{R} has a pure dipole form.

6. Multipole decomposition

A concrete evaluation of the gravitational self-force, for a particle moving in the field of a Schwarzschild or Kerr black hole, requires a multipole decomposition of the retarded perturbation,

$$h(x) = \sum_\ell h_\ell(x) \qquad \text{(sum diverges when } x = z\text{)}$$

Because the singular field is known analytically in the vicinity of the world line, it too can be expressed as a mode sum,

$$h^{\mathrm{S}}(x) = \sum_\ell h^{\mathrm{S}}_\ell(x) \qquad \text{(sum diverges when } x = z\text{)}$$

The multipole decomposition of the radiative field is

$$h^{\mathrm{R}}(x) = \sum_\ell h^{\mathrm{R}}_\ell(x) = \sum_\ell \Big[h_\ell(x) - h^{\mathrm{S}}_\ell(x) \Big]$$

This sum converges everywhere.

The self-force is evaluated as

$$\begin{aligned} F_{\text{self}} &= m \lim_{x\to z} \nabla h^{\text{R}} \\ &= m \lim_{x\to z} \sum_\ell \Big[(\nabla h)_\ell - (\nabla h^{\text{S}})_\ell \Big] \\ &= m \sum_\ell \lim_{x\to z} \Big[(\nabla h)_\ell - (\nabla h^{\text{S}})_\ell \Big] \\ &= \sum_\ell \Big[F_\ell^{\text{bare}} - F_\ell^{\text{S}} \Big] \end{aligned}$$

where

$$F_\ell^{\text{bare}} = m \lim_{x\to z} (\nabla h)_\ell$$

is constructed from the modes of the retarded field $\nabla_\gamma h_{\alpha\beta}$, and

$$F_\ell^{\text{S}} = m \lim_{x\to z} (\nabla h^{\text{S}})_\ell$$

is constructed from the modes of the singular field $\nabla_\gamma h^{\text{S}}_{\alpha\beta}$.

The multipole decomposition of the singular field has been worked out for Schwarzschild spacetime [Barack, Mino, Nakano, Ori, and Sasaki (2002); Detweiler, Messaritaki, and Whiting (2003); Kim (2004)].

It also has been worked out for Kerr spacetime [Barack and Ori (2003)].

The general structure is

$$F_\ell^{\mathrm{S}} = \big(\ell + \tfrac{1}{2}\big) A + B + \frac{C}{\ell + \frac{1}{2}} + \text{convergent terms}$$

The **regularization parameters** A^μ, B^μ, and $C^\mu \equiv 0$ are independent of ℓ but they depend on the spacetime and the particle's state of motion.

To illustrate, the mode decomposition of the **singular Newtonian potential** is given by

$$\Phi_{\rm S} = -\frac{m}{|\boldsymbol{x}-\boldsymbol{R}|} = -m\sum_\ell \frac{(r_<)^\ell}{(r_>)^{\ell+1}} P_\ell(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{R}})$$

where $r_< = \min(r,R)$, $r_> = \max(r,R)$, and $\hat{\boldsymbol{n}} = \boldsymbol{x}/r$.

Taking a gradient and then the limit $\boldsymbol{x} \to \boldsymbol{R}$ produces

$$F^{{\rm S}\,r}_\ell = \frac{m^2}{R^2}\left[\mp\left(\ell+\tfrac{1}{2}\right) - \frac{1}{2}\right]$$

where the limit is taken as $r \to R^\pm$.

The angular components of $\boldsymbol{F}^{\rm S}_\ell$ vanish.

This result indicates that $A_r = \mp m^2/R^2$, $B_r = -m^2/(2R^2)$ in the Newtonian limit, which is also predicted by the relativistic results.

Computation of the self-force for **Schwarzschild spacetime** is now in principle straightforward; results should be available soon [Barack and Lousto (2002); Barack and Lousto; Hikida, Nakano, Tanaka, and Sasaki; Detweiler and Whiting].

A calculation for **Kerr spacetime** is less so, because $h_{\alpha\beta}$ must first be recovered from the Teukolsky curvature variables [Chrzanowski (1975); Wald (1978); Lousto and Whiting (2002); Ori (2003); Mino (2003)].

The self-force was computed for **weakly-curved spacetimes** and shown to agree with standard post-Newtonian theory [Pfenning and Poisson (2002)].

7. Beyond the self-force

The gravitational self-force is **not gauge invariant** — it is formulated in the Lorenz gauge ${\psi^{\alpha\beta}}_{;\beta} = 0$ — and it must be combined with other inputs to extract observational consequences.

For example, if the particle is a pulsar, then the self-force and the perturbed metric $\mathsf{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$ can be used together to calculate the pulses' times-of-arrival.

Ultimately we want gravitational waveforms, and this requires inputs from **second-order perturbation theory**.

Consider the first-order problem:

$$(\Box + 2R)h_1 = -16\pi T[z]$$

Consistency demands that $z(\tau)$ be a geodesic of the background spacetime, and the waveforms obtained from h_1 do not incorporate self-force information.

Consider next the second-order problem:

$$(\Box + 2R)h_2 = -16\pi(1 + h_1)T[z + \delta z] + (\nabla h_1)^2$$

where $\delta z(\tau)$ is the correction to the geodesic motion.

The waveforms obtained from h_2 properly incorporate self-force information, in a gauge-invariant manner.

Getting waveforms therefore requires (defining and) solving the second-order problem [Ori and Rosenthal].

7. Conclusion

There has been significant progress in the self-force problem over the last several years:

- the foundations are solid;
- the regularization parameters A^μ, B^μ, and C^μ are known for arbitrary motion in Schwarzschild and Kerr spacetimes;
- metric reconstruction from the Teukolsky curvature variables is better understood.

But there are difficult outstanding issues:

- how to define and calculate the low-multipole ($\ell = 0$ and $\ell = 1$) contributions to the self-force in Kerr?
- what useful gauge-invariant quantities can be computed from the self-force and the metric perturbation?
- how to properly incorporate the self-force into a wave-generation formalism?