August 7, 2007

NUMERICAL SOLUTION OF LAPLACE'S EQUATION

PURPOSE:

This experiment illustrates the numerical solution of Laplace's Equation using a relaxation method. The results of the relaxation method are compared with the exact analytical solutions so that conclusions may be drawn about the precision and convergence of the numerical method. The relaxation method may be programmed and run on a computer using any programming language; alternatively, one may program a spreadsheet. The examples and exercises that follow assume the spreadsheet will be used.

REFERENCE: *Introduction to Electrodynamics*, D. J. Griffiths; pg. 111-114 & pg. 127-135
T.T. Crow, Am. J. Phys. 55, 817 (1987).

INTRODUCTION:

Laplace's Equation arises in areas of physics including steady-state heat flow, and in free space, the gravitational potential, the electrostatic and the magnetic potential. In the electrostatic case, we begin with Gauss's Law, which, in differential form, is

$$\vec{\nabla} \bullet \vec{E} = \rho / \varepsilon_o \quad , \tag{1}$$

where ρ is the charge density and ε_o the permittivity of free space. The electric field $\vec{E}$ and the potential V are related by

$$\vec{E} = - \vec{\nabla} V \quad . \tag{2}$$

Consequently, combining (1) and (2) gives

$$\nabla^2 V = - \rho / \varepsilon_o \quad , \tag{3}$$

which is known as Poisson's Equation. In charge-free regions, $\rho = 0$, and consequently,

$$\nabla^2 V = 0 \quad . \tag{4}$$

This equation is known as Laplace's Equation. In this experiment, we will confine ourselves to a maximum of two dimensions, so that in Cartesian co-ordinates, (4) becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad , \tag{5}$$

and in cylindrical co-ordinates, Eq.(4) becomes

$$\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} = 0 \quad , \tag{6}$$

where it is assumed that the potential is not a function of Θ.

Equations (5) and (6) are partial differential equations, and by themselves, do not result in unique solutions of V. In order to determine the solution explicitly we require an acceptable set of boundary conditions (i.e., the value of V or its derivative at all points on the boundaries). The solutions to Laplace's equation have the property that the value of V at any point (x,y) is the average value of V on a circle, of any radius, about the point (x,y). In one dimension, the value of V(x) is equal to $^1/_2$ [V(x + Δx) + V(x - Δx)] for any Δx. As a consequence, the solutions to Laplace's equation (in both one and two dimensions) have no maxima or minima. Extreme values of V occur only at the boundaries.

The properties of these solutions suggest a simple way to generate a numerical solution to Laplace's equation. The process is known as the relaxation method. One starts by fixing the values of V at the boundaries and then reassigning the value of each interior point to the average value of its nearest neighbours. This process is repeated, using the values from the previous calculations to generate the next approximation. The iterations are repeated until the changes that occur to the values of the interior points are negligible.

To see how these properties may be used to obtain approximate numerical solutions to Laplace's equation, consider the simplest approximations to the first and second derivatives (obtained using a Taylor's series expansion) namely,

$$f(x \pm \Delta x) = f(x) \pm \Delta x\, f'(x) + \tfrac{1}{2}(\Delta x)^2 f''(x) \pm \ldots \tag{7}$$

Solving for the first derivative $f'(x)$ gives:

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2(\Delta x)} + \text{higher order terms} \tag{8}$$

and the second derivative $f''(x)$ is:

$$f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} + \text{higher order terms} \tag{9}$$

If Δx is small, then the expressions involving the higher order terms in (8) and (9) are negligible, and (8) and (9) are therefore useful approximations for the first and second derivatives. These approximations are used in Laplace's Equation and give rise to a finite difference equation. That is, in Cartesian co-ordinates, Laplace's Equation is approximated by

$$\frac{V(x+\Delta x, y) - 2V(x,y) + V(x-\Delta x, y)}{(\Delta x)^2} + \frac{V(x, y+\Delta y) - 2V(x,y) + V(x, y-\Delta y)}{(\Delta y)^2} = 0 \tag{10}$$

and if Δx = Δy, then

$$V(x,y) = \tfrac{1}{4}\left[V(x+\Delta x, y) + V(x-\Delta x, y) + V(x, y+\Delta y) + V(x, y-\Delta y)\right] \tag{11}$$

Using the same approximations in cylindrical co-ordinates, and if we assume Δr = Δz, then

$$\begin{aligned} V(r,z) = {} & \tfrac{1}{4}\left[V(r+\Delta r, z) + V(r-\Delta r, z) + V(r, z+\Delta z) + V(r, z-\Delta z)\right] \\ & + \frac{\Delta r}{8r}\left[V(r+\Delta r, z) - V(r-\Delta r, z)\right] \end{aligned} \tag{12}$$

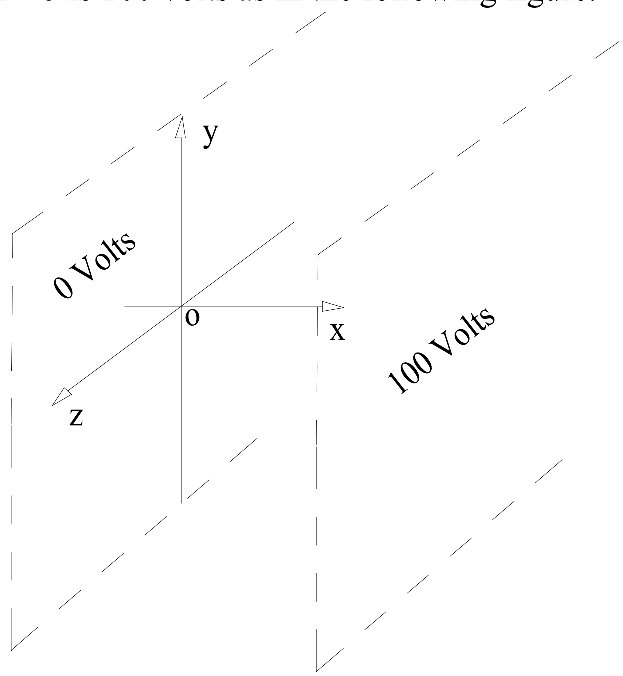
Eq.(11) states that the value of V at a point on a square grid is approximately the average of the values at the four nearest points. Eq.(12) is similar except that it applies to a system with cylindrical symmetry. The student should derive (fill in the missing steps) Eq.(11) and (12). The use of (11) and (12) is best illustrated via the following exercises.

NOTE: The following instructions apply for the use of Quattro Pro 9. Consult the teaching assistant if you encounter difficulties in locating the corresponding menu items in more recent versions of Quattro Pro. When saving your work to a USB flash drive, be mindful of the software you have available to you at home. If you only have access to Microsoft Excel at home, use the Quattro Pro "Save As" option to record your spreadsheet in an Excel format, particularly if you do not have printing capabilities in the CPES microcomputer lab. NOTE:

data saved to the hard drive of the computer will be deleted upon logging off! Please make sure you bring a USB flash drive or temporarily save to the Desktop and then email the file to yourself before logging off.

Exercise 1: **Infinite Parallel Plate Capacitor**

In this exercise we assume the plates have infinite extent in both the y and z direction so that the field lines are in the x direction. The space between the capacitor plates is arbitrarily divided into 4 units along the x axis. The potential of the plate at x = 0 is 0 volts and the plate at x = 5 is 100 volts as in the following figure.



This is a one-dimensional problem and the form of Laplace's equation, in the interior, is

$$\frac{d^2 V}{dx^2} = 0 \qquad (13)$$

subject to the boundary conditions V(0) = 0 and V(5) = 100. The analytic solution to this problem is trivial: V(x) = 20x.

For an approximate solution using the Taylor's series expansion approach, we substitute equation 9 into equation 13, neglecting higher order terms. This gives:

$$\frac{V(x+\Delta x)-2V(x)+V(x-\Delta x)}{(\Delta x)^2}=0$$

Solving for *V(x)*, we get:

$$V(x)=0.5\left[V(x+\Delta x)+V(x-\Delta x)\right] \tag{14}$$

We will solve Laplace's Equation using the relaxation method using a spreadsheet (specifically Quattro-Pro). We can then compare our results with the analytic solution for this simple problem. NOTE: Other software packages such as Lotus 123 and Microsoft Excel will also work. We use Quattro Pro here because it handles the manual self-referencing calculations more transparently than Excel does, despite the strong similarities between the two programs.

1) *Setting up the position co-ordinates and boundary conditions.*

Load Quattro Pro X3 from the WordPerfect Office X3 folder. In the electronic spreadsheet, each cell is identified by two co-ordinates, i.e., the column letter and the row number (as shown below). Row 1 identifies the n cells (n=6) that correspond to the position, x. Type in the values (0 - 5) for position in row one. In row 2, enter the potential of the capacitor plates (0 in A2, and 100 in F2).

		A	B	C	D	E	F
x	1	0	1	2	3	4	5
V(x)	2	0					100

2) *Changing the Calculation properties of your Spreadsheet.*

In the relaxation method, the potential in each cell depends on the potential of its neighbours. For example, the potential at $x = 1$ depends on that at $x = 2$. But the potential at $x = 2$ depends on that at $x = 1$. In automatic recalculation mode, Quattro Pro will stop calculating when it detects such a circular reference. We, however, want it to continue calculating, and stop after a certain number of iterations.

To do this, go to the Format menu, and select the Notebook... entry. In the dialog box that pops up, choose the Recalc Settings tab, and set Mode to Manual and the # of Iterations to 1. Finally, click OK.

3) *Calculate the first approximation for the potential using the relaxation method.*

Eq.(14) is applied to calculate the potential in each of the cells between the plates. In cell E2, type `=(F2 + D2)/2`. The result is V(E2) = 0.5[100 + 0] = 50. Notice that when you click on a cell you will see the formula or numerical value contained in the cell, in a small input line above the title of your work sheet. If you click on the input line you can edit the equation or value from within the input line or alternately, it may be edited from within the cell itself.

The next step is to copy the formula just entered to cells to the left of E2. Select the cell E2 (left click the mouse in it). A black box will appear around it. Use Ctrl + c to copy this expression to the clip board. Paste the expression into cell D2 (Ctrl +v), closing the dialog box that pops up. The resulting potential is V(D2) = 0.5[50 + 0] = 25. Use Ctrl + v to paste the expression into each of the remaining empty cells in row 2. The values should be V(C2) = 12.5 and V(B2) = 6.25. This is the first approximation and an example result is shown below; the actual values depend on the order the formulae were pasted into the cells. Save your work to your own USB flash drive or network workspace (use Save As).

		A	B	C	D	E	F
x	1	0	1	2	3	4	5
V(x)	2	0	6.25	12.5	25	50	100

4) *Recalculating the spreadsheet and showing the final result.*

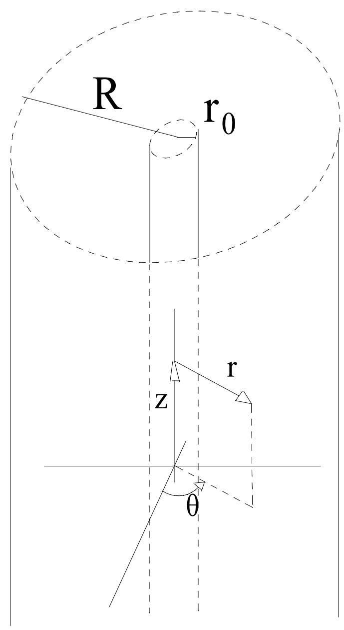
Press F9 to recalculate the spreadsheet. Notice the change. Press F9 four more times and record this result.

5) *Change the number of iterations.*

Through the Format – Notebook menu item, change the number of iterations to 45 in the Recalc Settings. Use the function key F9 to recalculate the notebook. After approximately 50 iterations, the relaxation method will approach the theoretical values with insignificant deviation. Your report should contain the first approximation, the result after an intermediate number of iterations (5) and the values to which the relaxation method converges. Also include (for reference) the theoretical values from the analytical solution of potential at each of the equally spaced points (cells) between the plates i.e., V(4) = 80, V(3) = 60 and so on.

Exercise 2: **Infinite Cylindrical Capacitor**

We assume that the capacitor is infinite in the z direction, and that the outer electrode is at a potential of 100 volts and of radius 10 units, while the inner electrode is at 0 volts and of radius 1 unit as in the following figure.



Because of the symmetry of the problem, the potential within the capacitor, Laplace's equation, has only r dependence. In this case Eq. (6) reduces to

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} = 0 \qquad (15)$$

where V(r) is subject to the boundary conditions V(1) = 0 and V(10) = 100. Using Gauss's law the student should show that the theoretical solution to this problem is given by:

$$V(r) = \frac{100}{\ln\left(\frac{R}{r_o}\right)} \ln\left(\frac{r}{r_o}\right) \tag{16}$$

where r_o and R are the radii of the inner and outer electrodes respectively.

For this case, the discrete form of Laplace's Equation, Eq. (12), becomes

$$V(r) = 0.5\left[V(r+\Delta r) + V(r-\Delta r)\right] + \frac{\Delta r}{4r}\left[V(r+\Delta r) - V(r-\Delta r)\right] \tag{17}$$

Eq.(17) may be rewritten in a form suitable for the relaxation method as follows:

$$V(r) = 0.5\left[V(r-\Delta r)\cdot\left(1-0.5\left(\frac{\Delta r}{r}\right)\right) + V(r+\Delta r)\cdot\left(1+0.5\left(\frac{\Delta r}{r}\right)\right)\right] \tag{18}$$

In a new spreadsheet, enter the cell numbers (row 1) and the boundary potentials (row 2) as shown below. In this exercise the cell size should be approximately 1/2 the size chosen in Exercise 1. Ensure that the Recalc settings has Mode set to Manual, as you did in Exercise 1.

		A	B	C	D	E	F	G	H	I	J
x	1	1	2	3	4	5	6	7	8	9	10
V(r)	2	0									100

1) *Calculating the first approximation to the potential using the relaxation method for the cylindrical capacitor problem.*

Eq.(18) gives the method to be applied to calculate the potential in a particular cell (position r between the plates). Enter the following formula into cell I2: `=(H2*(1 - 0.5/I$1) + J2*(1 + 0.5/I$1))/2`. The dollar sign in the I$1 term means that row 1 is fixed but the column letter is free to change relative to the current position (*i.e.,* where the formula is pasted). Copy and paste the formula from cell I2 into each of the empty cells in row 2. If you've done this correctly then your spreadsheet will appear as shown below, to two decimal places.

		A	B	C	D	E	F	G	H	I	J
r	1	1	2	3	4	5	6	7	8	9	10
V(r)	2	0	0.92	1.47	2.52	4.47	8.14	15.02	28.04	52.8	100
$V_{th}(r)$	3	0	30.1	47.7	60.2	69.9	77.8	84.5	90.3	95.4	100

2) *Using mathematical functions in Quattro Pro to calculate the exact theoretical solution and changing the numerical format.*

Row 3 of the table above shows the exact values calculated from the theoretical solution, Eq. (16). Enter the following formula into cell J3: `=@ln(J1)/@ln(10)*100`. The @ sign is used before all mathematical functions in Quattro Pro. Be sure you understand this formula. Copy this formula to the other cells in the row, A3 to I3.

3) *Change the number of significant figures displayed.*

Click and hold the left mouse button in cell A1 and drag the mouse to highlight the range from A1 to J3. Select Format, Selection … Select the Numeric Format tab in the window that opens, and then select "Number" under the Numeric Formats list. Ensure that the number of decimal places is 2 before selecting Ok.

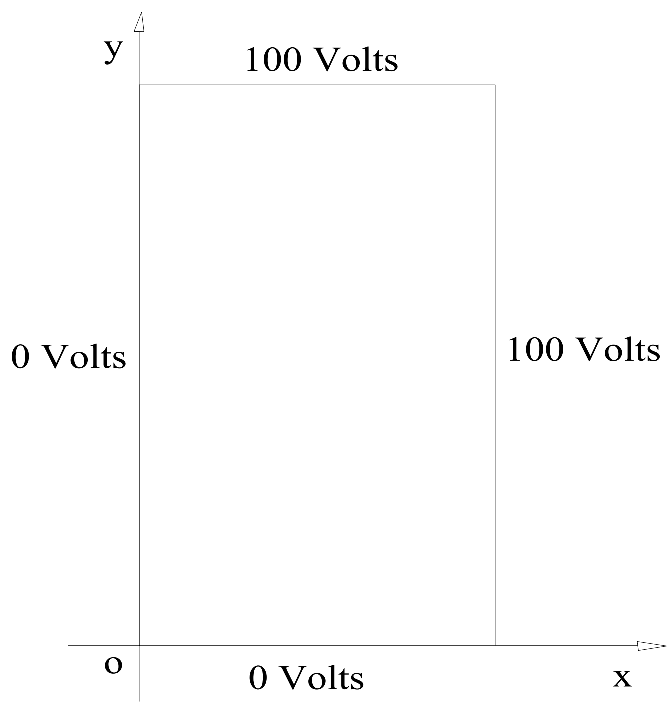
4) *Change the number of iterations to 50 and recalculate the notebook as in the previous exercise*

Record your result after 50 iterations and again after approximately 150 iterations. Additional iterations will cause negligible changes to the result.

Unlike the first exercise these values will not converge exactly with the theoretical values. Why do you think this is so? Specifically, why is convergence better for larger r? Make reference to the form of the theoretical solution as compared with the discrete form of Laplace's equation (Eq.(17)). In exercise 1, why is the discrete form of Laplace's equation (Eq. (14)) a better approximation to the linear theoretical solution (V(x) = 20x)? As in the previous exercise your report should contain the first approximation, the results after an intermediate number of iterations and the final values after 150 iterations.

Exercise 3: **A Two Dimensional Boundary Value Problem**

Consider a rectangle with a width of 10 units in the x direction, and a height of 20 units in the y direction. The potentials on the top and right boundaries are 100V, the left side and the bottom are at 0 volts as in the following figure.



This is a two dimensional problem and the form of Laplace's equation is

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0, \tag{21}$$

Subject to the boundary conditions V(x,0) = 0 for $0 \le x < 10$, V(x,20) = 100 for $0 \le x \le 10$, V(0,y) = 0 for $0 \le y < 20$, and V(10,y) = 100 for $0 \le y < 20$. The analytical solution of this problem is (it is not necessary to derive this):

$$V(x,y) = \lim_{N\to\infty} \frac{4}{\pi} V_o \sum_{n\ odd}^{N} \left(\frac{\sin(k_1 x)\sinh(k_1 y)}{n \sinh(k_1 b)} + \frac{\sin(k_2 y)\sinh(k_2 x)}{n \sinh(k_2 a)} \right) \qquad (22)$$

where

$$k_1 = \frac{n\pi}{a}, \qquad k_2 = \frac{n\pi}{b}, \qquad a = 10, \qquad b = 20 \quad and \quad V_o = 100$$

1) *Using the Spreadsheet to calculate the numerical solution for a two dimensional boundary problem.*

Ensure that the Recalc settings has Mode set to Manual, as you did in the previous exercises. Enter the boundary values as shown in the table below. The numerical solution is obtained by using Eq.(11) to calculate the potential at each of the interior points. Enter the following equation into cell B2: `=0.25*(B1 + C2 + B3 + A2).`

	A	B	C	D	E	F	G	H	I	J	K
1	100	100	100	100	100	100	100	100	100	100	100
2	0										100
3	0										100
.	.										.
.	.										.
.	.										.
21	0	0		. . .				0	0		100

Copy and paste this formula into the empty cells in the spreadsheet: Ctrl+c in cell B2, then highlight the empty cells below B2 in column B and press Ctrl+v. Then highlight the empty cells in columns C – J and press Ctrl+v. Save this spreadsheet. In the above table, the (x,y) co-ordinates of the cells A1, K1, K21, A21 are (0, 20), (10, 20), (10,0) and (0, 0) respectively, *i.e.*, the origin has been placed at the lower left corner.

2) *Set the number of iterations to 24 and recalculate the spreadsheet.*
Save this spreadsheet. Use Save As with a different name than used earlier.

3) Continue recalculating the spreadsheet until the changes are insignificant. Save the final result. Your report should contain a copy (printout) of your worksheet after the first approximation, a copy showing an intermediate number of iterations and a final copy showing the values to which the numerical method converge. Do not forget to record the number of iterations for each. You will need to comment on how each of the three data sets compares with the theoretical solution in your report. The next step will help you in calculating the theoretical solution based on equation (22).

4) For exercise 3, the theoretical result may be obtained fairly easily from within Quattro-Pro. To begin, set up a new spreadsheet as follows:

	A	B	C	D	E	F	G
1	V =	100					
2	a =	10					
3	b =	20					
4	x =	8					
5	y =	1					
6							
7	n	k1	k2	1st term	2nd term	Vn(x,y)	Vtot(x,y)
8	1	0.31416	0.15708	0.000701	0.109747	14.062	14.062
9	3	0.94248	0.47124	0.0000	0.058941	7.504	21.567
.	.						.
.	.						.
33	51						.

The equations for calculating k_1 and k_2 are given under equation (22). Enter these expressions in cells B8 to B33 and C8 to C33 respectively. In addition,

$$1^{st}\ \text{term} = \frac{\sin(k_1 x)\sinh(k_1 y)}{n\sinh(k_1 b)}$$

$$2^{\text{nd}}\ \text{term} = \frac{\sin(k_2 y)\sinh(k_2 x)}{n\sinh(k_2 a)}$$

$$V_n(x,y) = \frac{4V}{\pi}\left[(1^{\text{st}}\ \text{term})_n + (2^{\text{nd}}\ \text{term})_n\right]$$

and

$$V_{tot}(x,y) = \sum_{i=1}^{n} V_i(x,y)$$

The Quattro Pro function `@pi()` will be useful.
The initial entries in the table show the results for the case of x = 8, y = 1 for the first two lines. These results will change as the chosen integer values for x and y change. It is not expected that you will tabulate many of the 231 combinations of x and y; rather, do only a representative set of approximately 10 points to illustrate and compare the relaxation method and the theoretical result. The convergence of a data point on the boundary where V = 100 volts is of particular interest because the series solution oscillates about the value before finally converging. You may need to extend the range of n in your spreadsheet to see convergence on a boundary point. You should show this effect by tabulation and by graphing $V_{Tot}(x,y)$ as a function of n for an interior point and a point on the boundary. Why are more terms in the sum necessary for convergence on the boundary compared with interior points?

EXPECTATIONS FOR LAPLACE'S EQUATION FORMAL REPORT

- 10 pages maximum length
- Include an abstract

INTRODUCTION

- State the purpose of this experiment
- State some background
- Derive equations 8, 9, 11, 12 and 16

PROCEDURE

- State what was done, no recipes
- Describe all three cases

RESULTS AND ANALYSIS

1: Parallel Plate

- Give the first, intermediate and converged values with the number of iterations.
- State the analytical solution.
- Do the analytical values agree with the numerical results?

2: Cylindrical Capacitor

- State the first, intermediate and converged values with the number of iterations.
- State the theoretical values with % difference.
- State the relaxation formula for this case.

3: 2-D Problem

- Give the first, intermediate and converged values with the number of iterations.
- Give the full results for two points. Pick an interior point and a point next to the boundary. Plot V_{tot} (x,y) vs n for both points; why is the convergence slower on the boundary? Explain the oscillations on the boundary.
- Give % differences between theoretical and converged values for all 10 locations investigated.

DISCUSSION

- Why do we get exact results from the relaxation method for the parallel plate problem, yet only approximate results from the relaxation method for the cylindrical capacitor problem? (Explain in words and in math)

CONCLUSION

- Does the relaxation method work?
- Does the potential follow the characteristics of Laplace's equation?
- List possible improvements.
- Discuss systematics.