The Packed Swiss Cheese Cosmology and Multifractal Large-Scale Structure in the Universe

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What is a Fractal?

• A fractal is a set whose Hausdorff Dimension strictly exceeds its topological dimension

• An object whose parts somehow resemble the whole

• Self-similar; scale-invariant

• Recursive; infinite structure

\[ N(d) \propto d^{D_F} \]

\( D_F \) is the fractal dimension
Euclidean Geometry: Non-Fractal ($D_T = D_F$)

Point: $D_F = 0$

Disc: $D_F = 2.0$

Circle: $D_F = 1.0$

Sphere: $D_F = 3.0$

$N(d) \propto d^{D_F}$
How Do You Measure Fractal Dimensions?

• **Top-Down / Resolution-Refinement**
  – How “fast” can one approximate the structure of the fractal by increasing the “resolution” of measurement?

    Box counting, perimeter-length

• **Bottom-Up**
  – How does structure change for increasing distance (scales) away from an arbitrary point within?

    N-point correlation, conditional density, density reconstruction
Box Counting:
How does box count $n(d)$ vary with scale size?

Slope is $D_F$
Why Study Fractals?

- Can tell us something about the structure of the object
  "Fingerprinting" method

- Can tell us something about how it was constructed (can "reverse engineer" the formation)

- Good tool for early Universe "reconstruction" and understanding cosmological structure origins?
Fractal Nature of the Large Scale Structure of the Universe

- Cosmological Principle requires homogeneity and isotropy
  - Must look the same and act the same everywhere!

\[ N_{\text{galaxies}} \propto r^3 \Rightarrow D_F = 3 \]

- Equal probability of finding a galaxy in every direction

- Observation suggests otherwise! *Not H & I!*
  - We see “clumps” of matter and voids of nothing
  - show \( D_F \approx 2 \) scaling [e.g. Pietronerro *et al.*] out to 1000 Mpc (?)

- Models of Universe are *based* on Cosmological Principle

*Which is right???*
Enter the Swiss Cheese Cosmology

- Can both be correct if the Universe has a “Swiss Cheese” structure

*Local inhomogeneities with global homogeneity*

- Not new [Einstein and Strauss (1945); Schücking (1954); Rees and Scieama (1968)]

Swiss Cheese Formalism

- Schwarzschild “hole” matched with FRW “cheese”

\[
\text{Hole} \quad ds^2 = \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right) dt^2 - \left[\frac{dr^2}{1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}} + r^2 d\Omega^2\right]
\]

\[
\text{FRW} \quad ds^2 = dt^2 - R^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right]
\]

- Vanishing of Weyl tensor on sphere surface ensures recovery of H&I (no tidal “compass”)

- Swiss Cheese cosmologies can be built in spaces of positive, negative, or flat curvature (k = +1,-1,0).
Swiss Cheese Packing

Spherical distribution of matter (dust)
Swiss Cheese Cosmology: First-Level Iteration

Inscribed surfaces

[Models contain 35,000 - 80,000 spheres]
2-D “sky” projection of sphere centers ("clumps")

Looks real!

3-D distribution of sphere centers ("clumps" of matter)

If SC agrees with observation: should show $D_F \approx 2$
Measuring the Fractal Dimension of “Packed Swiss Cheese” Via Standard Box Counting

SAME FOR $k = +1, 0, -1$

$D_F = 2.7 \pm 0.1$

$\neq 2.0$ !
The Conditional Average Density

- Box counting really doesn’t account for curvature
- We can calculate $D_F$ by tracing along geodesics on the manifold

\[
\Gamma(r) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{A_i(r)} \frac{dN_i(r)}{dr} \rightarrow r^{D-3}
\]

Estimates the average change in number of points within a spherical region of radius $r$, centered at an arbitrary point $i$ within the space.
Flat curvature

$D_F \sim 2.8$
Positive curvature

$D_F \sim 2.8$
Negative curvature

$D_F \sim 2.8$
So......

• Can’t explicitly determine geometry of Universe from fractal dimension

• No apparent match to observation

• Is fractal dimension a useless statistic?

• $D_F \sim 2.5$ is roughly fractal dimension of 3-dimensional Appolonian packing

• What else can we learn from it?
Not Just Fractals….

Multifractals!

- Object whose structure cannot be described by a single scaling behavior
- Union of an infinite number of fractals
- How to measure?
  - Modify Box Counting to account for local “density” of pattern in each cell
Natural Multifractals

• The world isn’t really “Euclidean”

• Many objects in Nature exhibit fractal-like structure; more details at higher magnifications

• *Statistical* self-similarity over a *finite* range of length scales

• Can study vegetation growth patterns, traffic flow (timeseries), topographical terrain, artwork
Multifractal Measure

- **Density**
  \[ p_i(d) = \frac{n_i(d)}{N_{tot}} \]

- **Measure**
  \[ Z(q, d) = \sum_{i=1}^{N(d)} [p_i(d)]^q \]

- **MF Dimension**
  \[ \tau(q) = \frac{d \log[Z(q, d)]}{d \log[d]} \]
  \[ D_q = \frac{\tau(q)}{q - 1} \]
Multifractal Dimension Spectrum

\[ D_{-\infty} \]

\[ D_0 = D_\infty \]

\[ q < 0 \]

\[ q > 0 \]

\[ q = 0 \]
What do the $D_q$ tell us?

- $q$ is a “filter” or “magnifying glass”
- $q > 0$ highlights dense portions of the pattern
- $q < 0$ highlights sparse portions of the set
- $q \to \infty$ shows *strongest clustering regions*
- $q \to -\infty$ shows *least dense regions*
Swiss Cheese Model

Random points

\[ D \approx 2.8 \]

\[ D = 3 \]

“Pseudo-linear”

Linear
Multifractal Spectra of Distributions

SC model can be used to identify formation mechanism.
Geodesic Considerations: Density Reconstruction Function

- As with regular fractal dimension, box counting to determine multifractal spectrum seems limiting.
- Can use the density reconstruction function to determine multifractal moments from a “bottom-up” approach (as with conditional density).
- Evaluate along geodesic distance from randomly selected point in catalog.

\[
W(\tau, q) = \frac{1}{N} \sum_{i=1}^{N} r_i(p)^{-\tau} \rightarrow p^{1-q}
\]
Density Reconstruction Multifractal Spectrum (Flat)

Rarest density scaling

Strongest density scaling
Density Reconstruction Multifractal Spectrum (Positive)
Density Reconstruction Multifractal Spectrum (Negative)
General Summary of Multifractal Results

<table>
<thead>
<tr>
<th></th>
<th>Swiss Cheese</th>
<th>Observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{-\infty}$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$D_0$</td>
<td>2.5-2.8</td>
<td>2</td>
</tr>
<tr>
<td>$D_{\infty}$</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

- No explicit match between theory and experiment
  - Different formation models
- $D_{\infty} = 2$; Strongest clustering occurs on *surfaces* for SC
- Observation suggests local clustering is $D = 2$; strongest clustering is *filamentary*, $D_{\infty} = 1$ [Filaments on sheets]

Even though they look the same, structure is different due to construction paradigm
Luminosity Basing Considerations

• Model isn’t doomed!

• If we account for luminosity biasing considerations, then fractal analysis methods can yield seemingly lower dimensions (sample-size dependence??)

• Assume $M(L) \sim L^\beta \ \beta \sim 1$, use mass cut-off roughly 0.01% that of largest in catalog

• SSRS data analysis suggests strong connection between statistical clustering behavior and luminosity, weighted toward brighter galaxies [Benoist et al., 1996], and earlier evidence of a mismatch between galaxy correlation functions w/r to cluster richness [Bahcall and Soneira. 1983]
Current Projects and Future Directions

- Hybrid model of PSC with “N-body” clustering on surfaces?

- Fractality and curvature
  - How does curvature affect the notion of fractal structure? [Dyer and Mureika, in preparation]
  - Traditional fractal dimension seems rooted in Euclidean definitions
  - Hilbert’s Congruence Axioms on curved manifolds?

- Lacunarity analysis of clustering
  - Voids instead of clumps!
  - Can curvature be detected this way?
Acknowledgments

This work is supported by a Postdoctoral Fellowship From the Natural Sciences and Engineering Research Council of Canada. Thanks to Charles Dyer (University of Toronto) for continued collaboration, and to Pitzer College for additional Financial support.

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