The Packed Swiss Cheese Cosmology and Multifractal Large-Scale Structure in the Universe

J. R. Mureika
W. M. Keck Science Center, The Claremont Colleges
Claremont, California

W.M. KECK Science Center

Claremont McKenna · Pitzer · Scripps
Members of The Claremont Colleges

What is a Fractal?

- A fractal is a set whose Hausdorff Dimension strictly exceeds its topological dimension
- An object whose parts somehow resemble the whole
- Self-similar; scale-invariant
- Recursive; infinite structure

$$N(d) \propto d^{D_F}$$

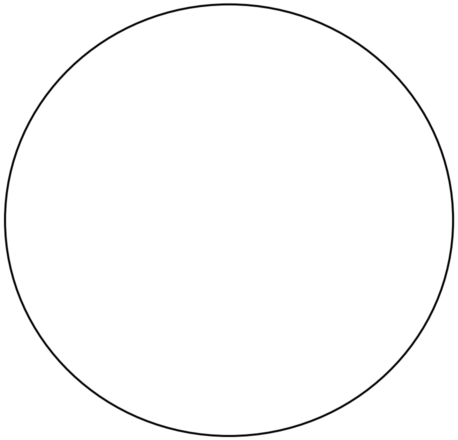
Characteristic → $N(d)$

Scale length → d

D_F is the *fractal dimension*

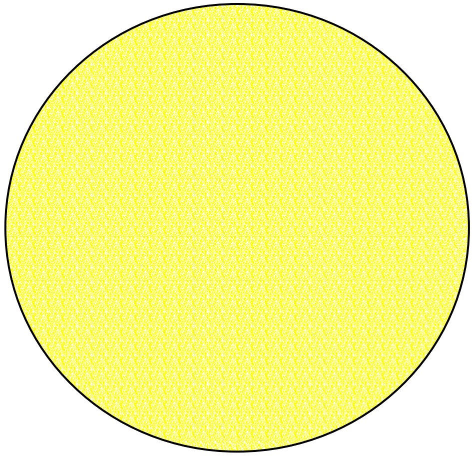
Euclidean Geometry: Non-Fractal ($D_T = D_F$)

Point: $D_F = 0$

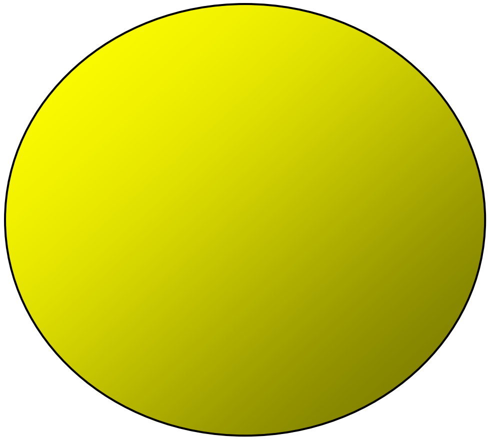


Circle: $D_F = 1.0$

$$N(d) \propto d^{D_F}$$



Disc: $D_F = 2.0$



Sphere: $D_F = 3.0$

How Do You Measure Fractal Dimensions?

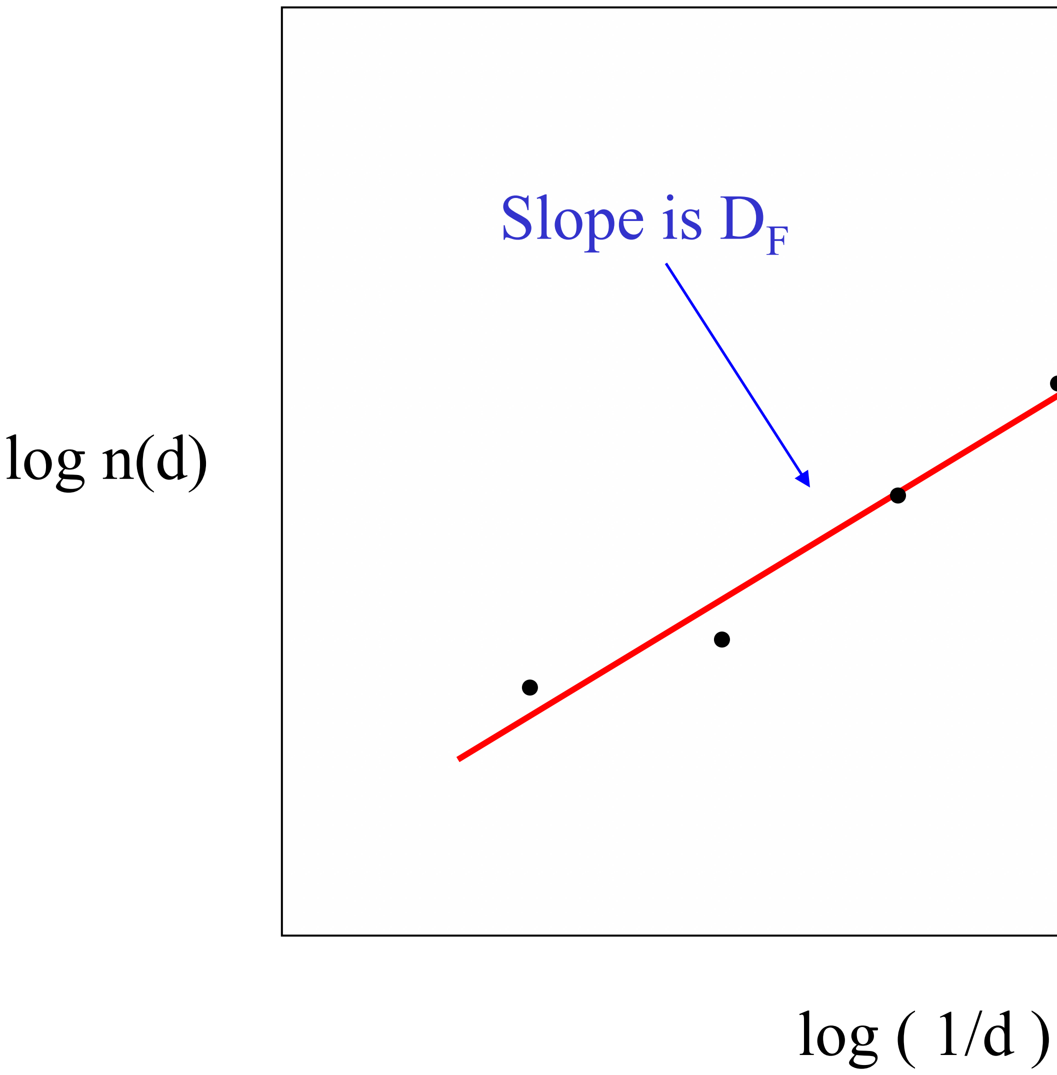
- **Top-Down / Resolution-Refinement**
  - How "fast" can one approximate the structure of the fractal by increasing the "resolution" of measurement?

    Box counting , perimeter-length

- **Bottom-Up**
  - How does structure change for increasing distance (scales) away from an arbitrary point within?

    N-point correlation, conditional density, density reconstruction

Box Counting:
How does box count n(d) vary with scale size?



Why Study Fractals?

- Can tell us something about the structure of the object

    "Fingerprinting" method

- Can tell us something about how it was constructed (can "reverse engineer" the formation)

- Good tool for early Universe "reconstruction" and understanding cosmological structure origins?

Fractal Nature of the Large Scale Structure of the Universe

- Cosmological Principle requires homogeneity and isotropy
  - Must look the same and act the same everywhere!

$$N_{galaxies} \propto r^3 \Rightarrow D_F = 3$$

- Equal probability of finding a galaxy in every direction
- Observation suggests otherwise! *Not H & I!*
  - We see "clumps" of matter and voids of nothing
  - show $D_F \approx 2$ scaling [e.g. Pietronerro *et al.*] out to 1000 Mpc (?)
- Models of Universe are *based* on Cosmological Principle

Which is right???

Enter the Swiss Cheese Cosmology

- Can both be correct if the Universe has a “Swiss Cheese” structure

Local inhomogeneities with *global* homogeneity

- Not new [Einstein and Strauss (1945); Schücking (1954); Rees and Scieama (1968)]

- Multiple SC cosmologies: Kantowski (1969), Dyer (1974), Dyer and IP (1988), Ribeiro (1992-1993), *etc*…

Swiss Cheese Formalism

- Schwarzschild “hole” matched with FRW “cheese”

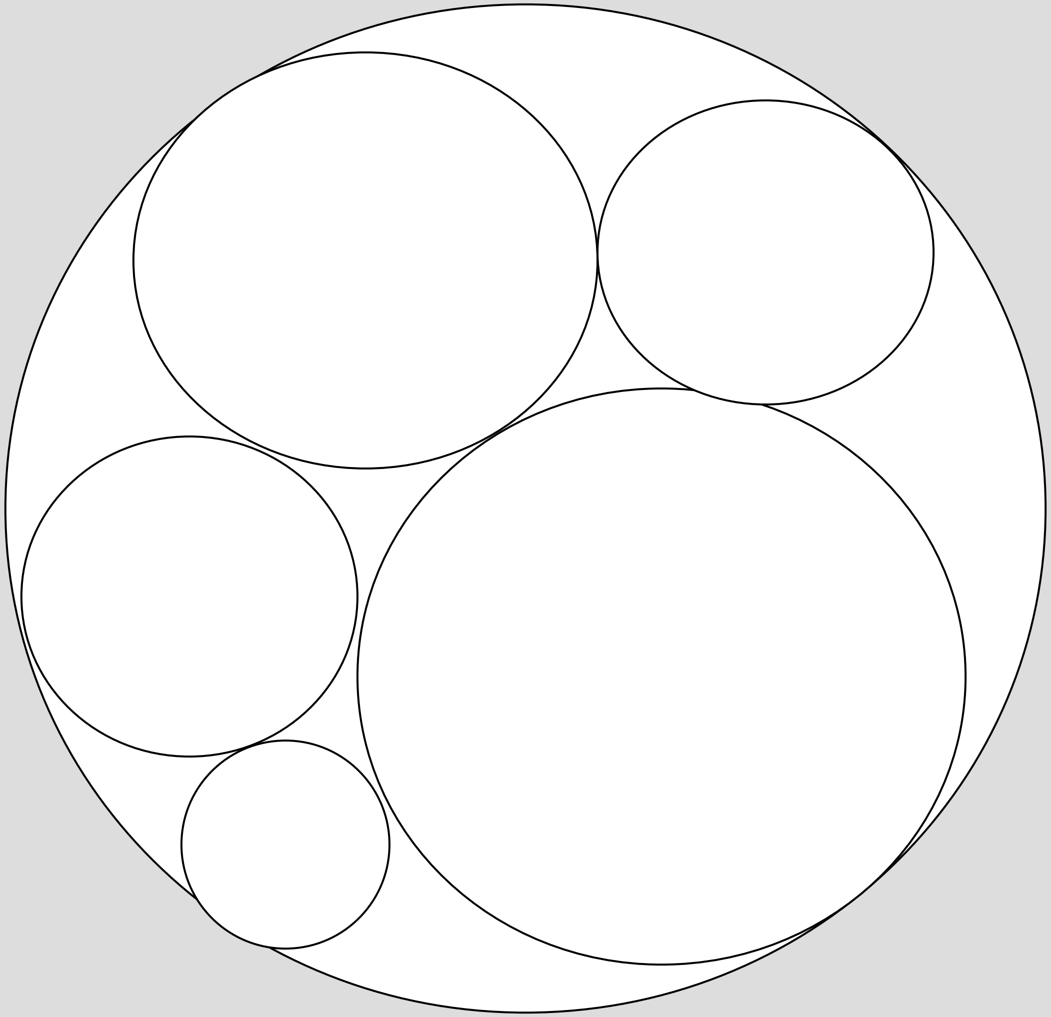
Hole $$ds^2 = (1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3})dt^2 - \left[\frac{dr^2}{1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}} + r^2 d\Omega^2\right]$$

FRW $$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right]$$

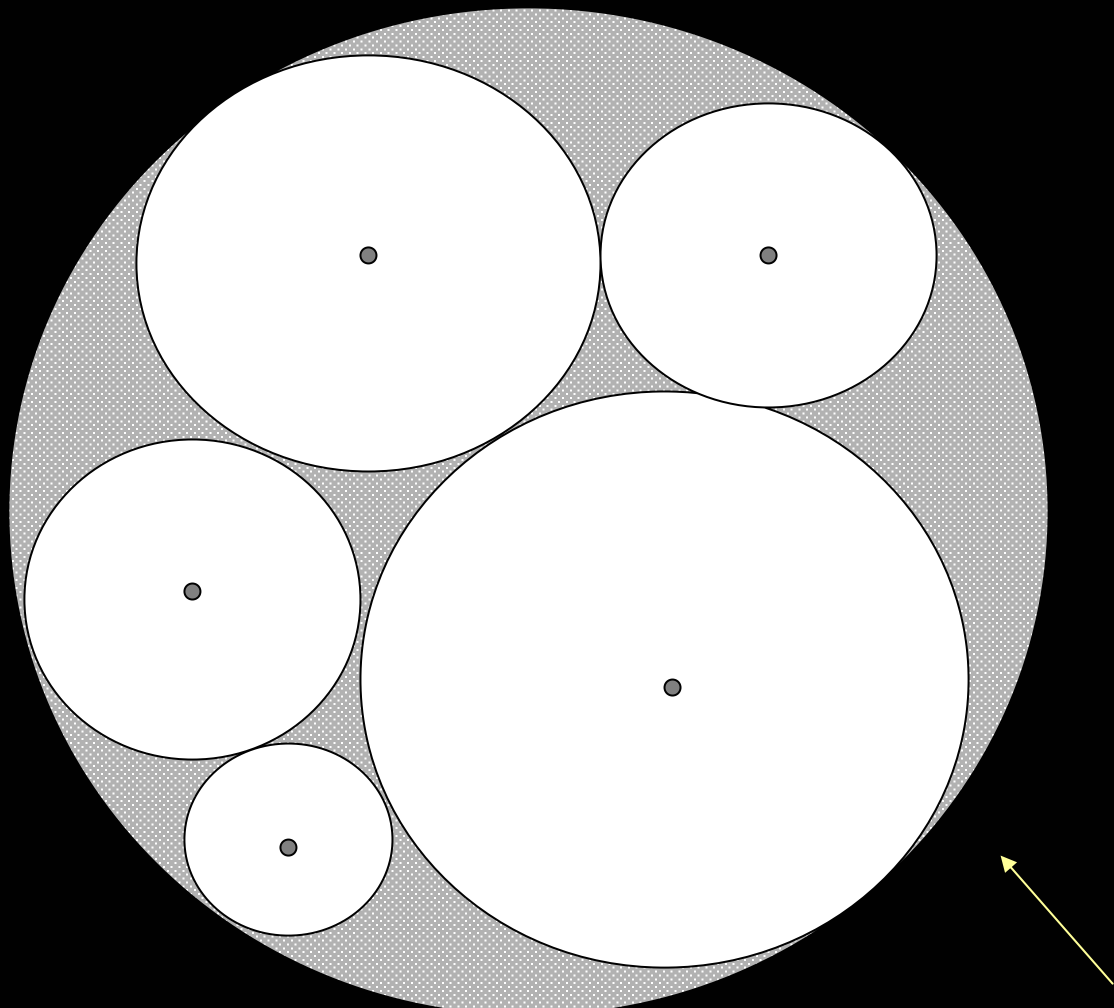
- Vanishing of Weyl tensor on sphere surface ensures recovery of H&I (no tidal “compass”)

- Swiss Cheese cosmologies can be built in spaces of positive, negative, or flat curvature (k = +1,-1,0).

Apollonian Packing

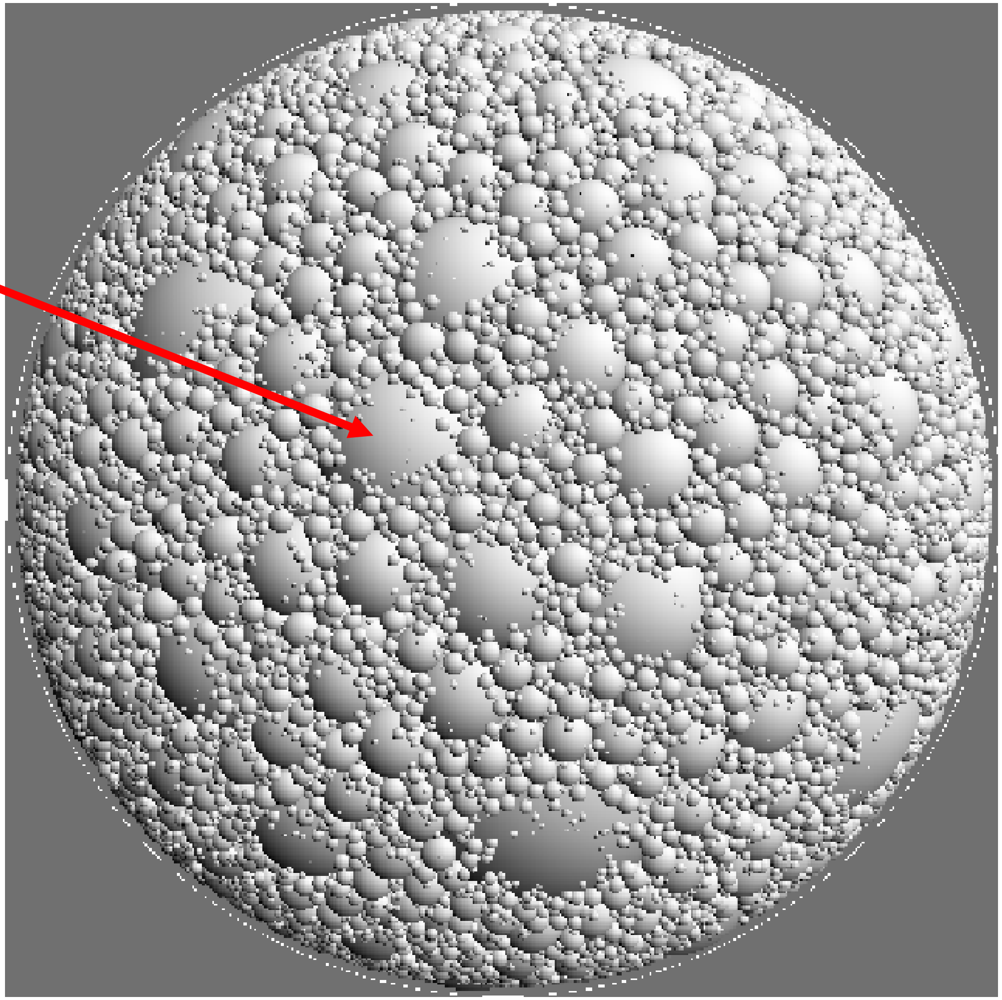


Swiss Cheese Packing

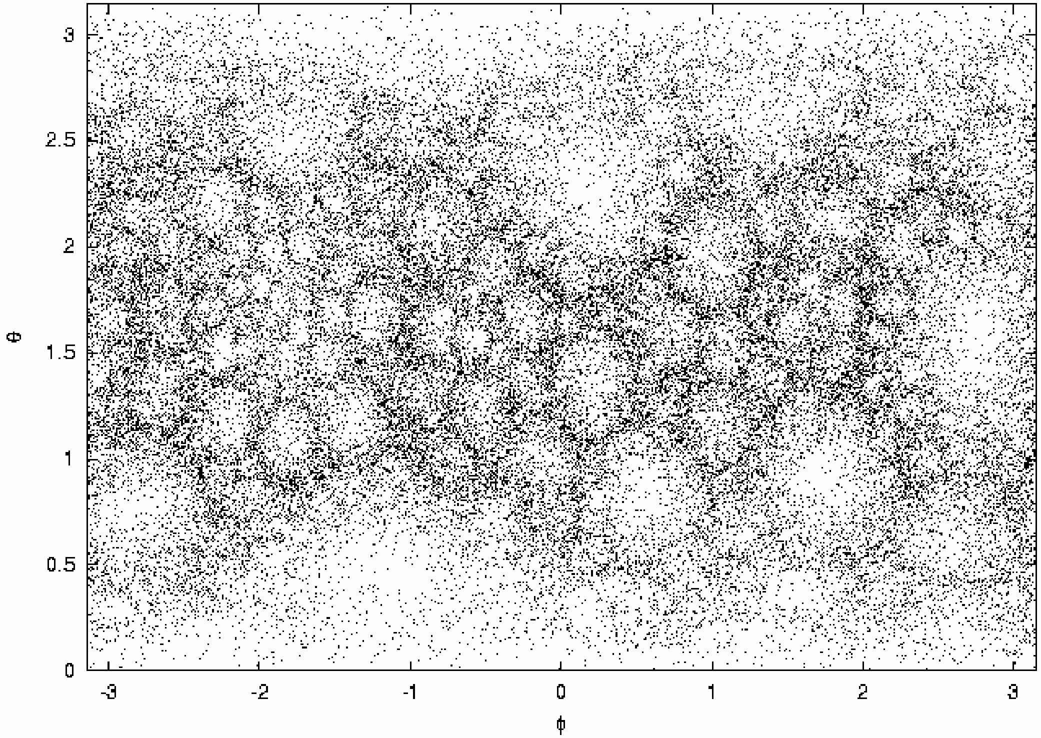


Swiss Cheese Cosmology: First-Level Iteration

Inscribed surfaces



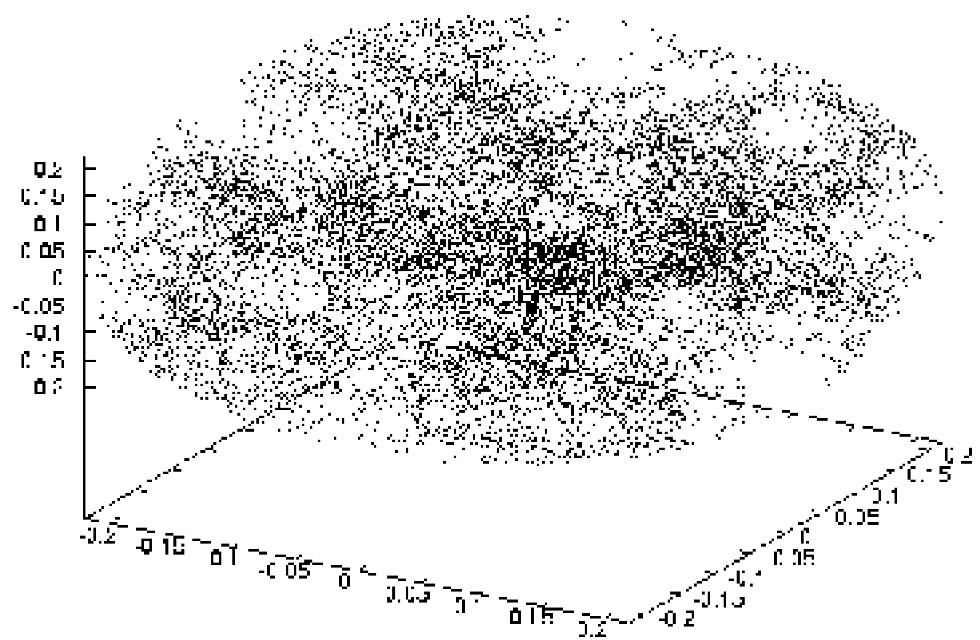
[Models contain **35,000 - 80,000** spheres]



2-D "sky" projection of sphere centers ("clumps")

Looks real!

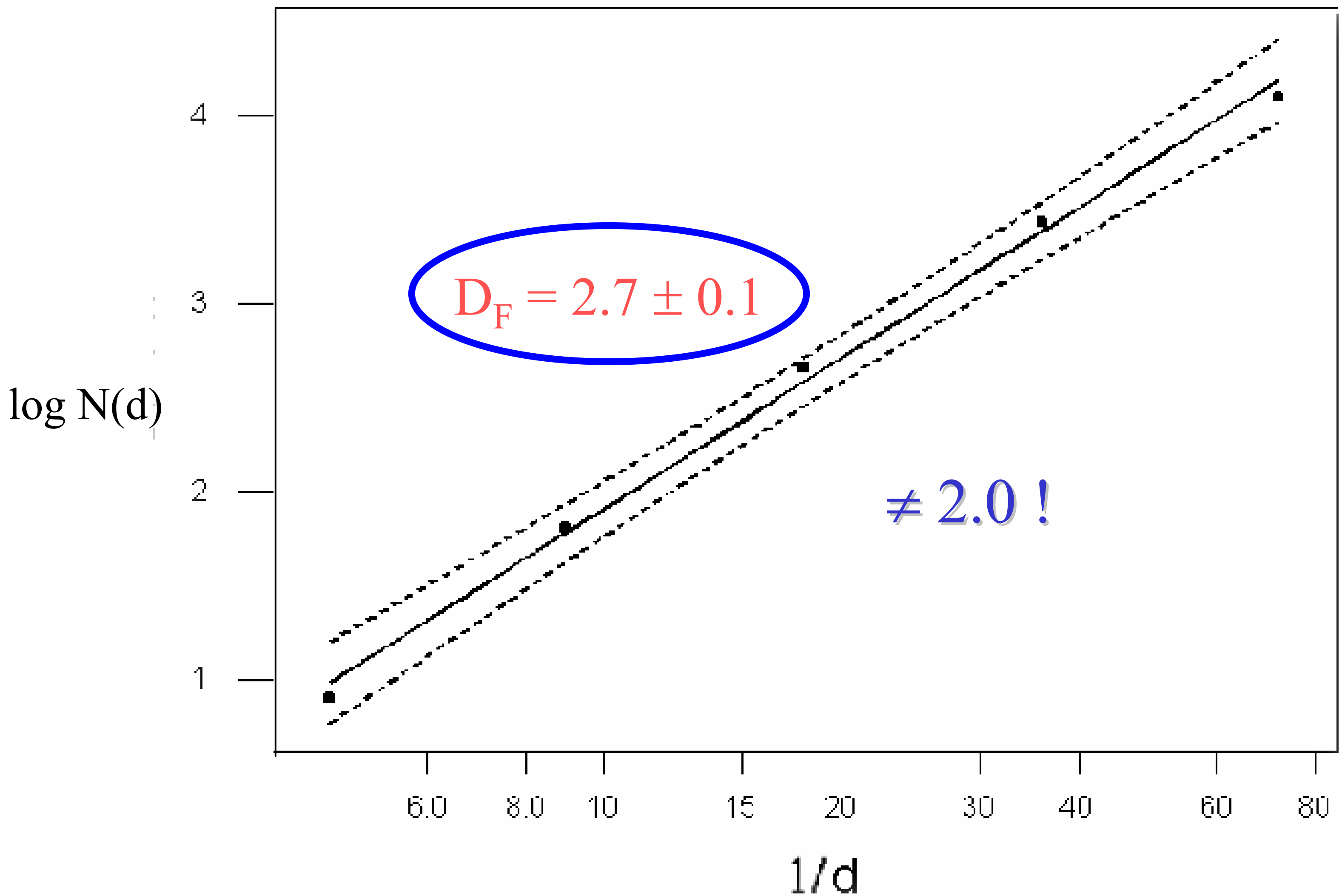
3-D distribution of sphere centers ("clumps" of matter)



If SC agrees with observation: should show $D_F \approx 2$

Measuring the Fractal Dimension of "Packed Swiss Cheese" Via Standard Box Counting

SAME FOR k = +1, 0, -1



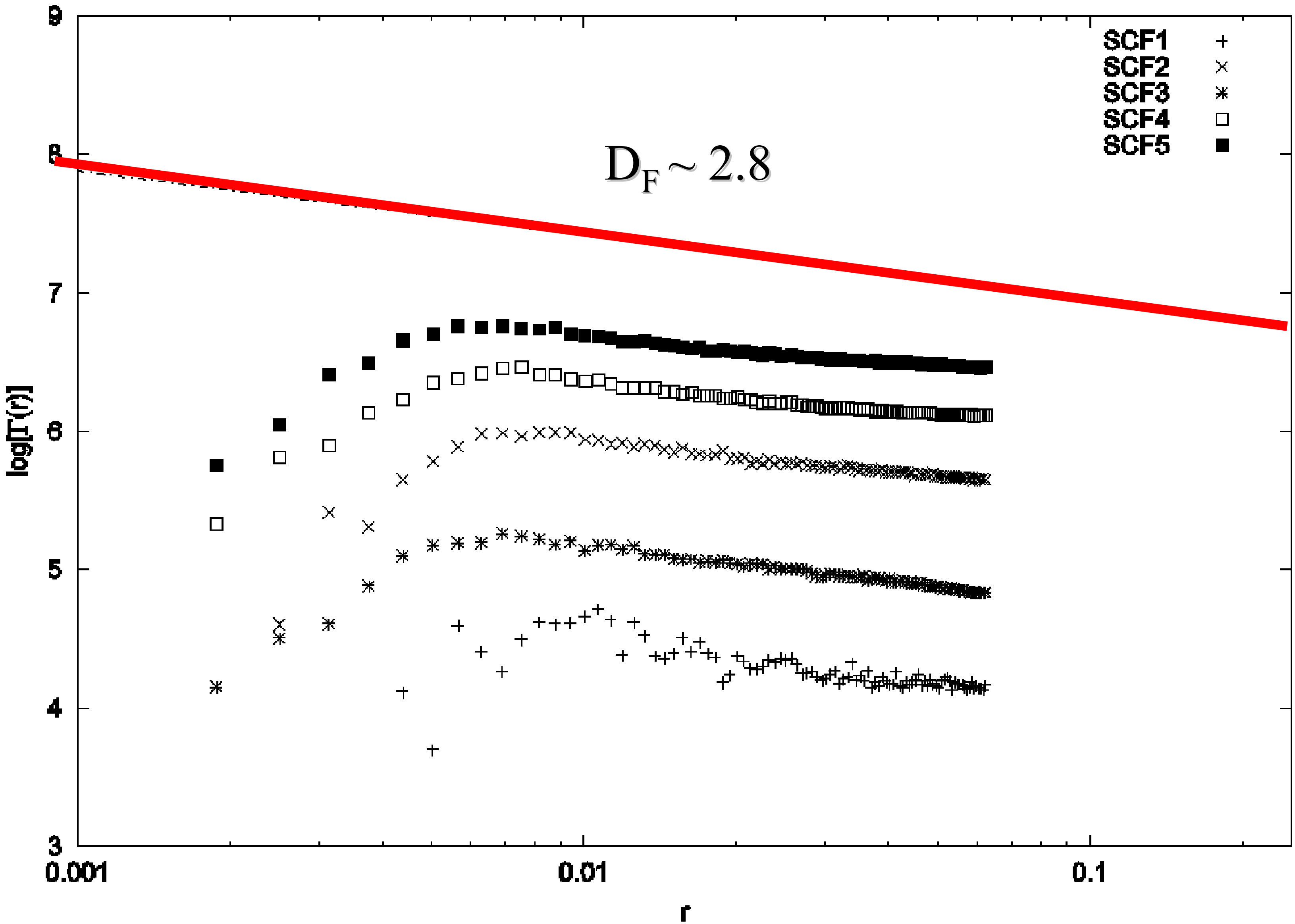
The Conditional Average Density

- Box counting really doesn't account for curvature
- We can calculate D_F by tracing along geodesics on the manifold

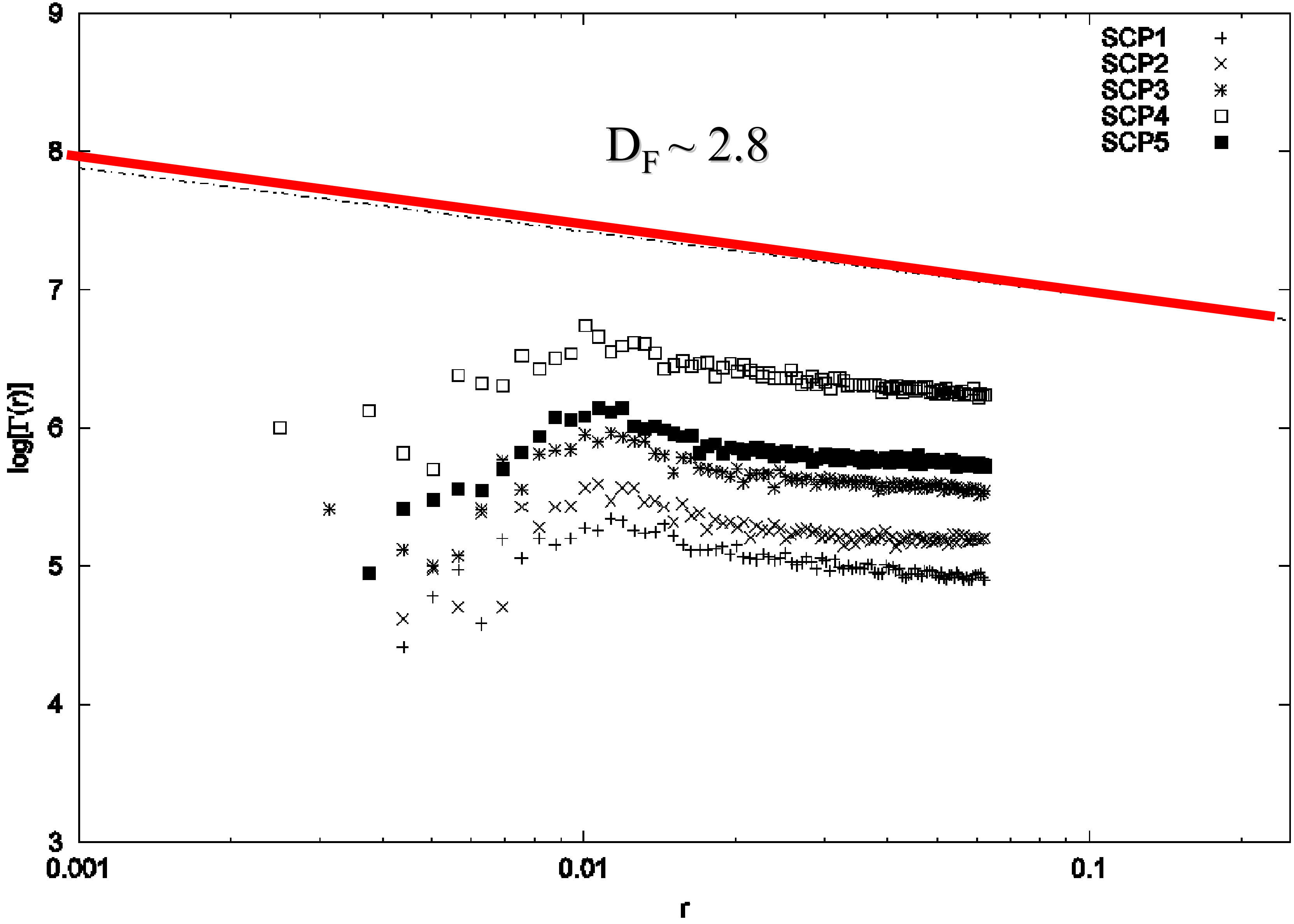
$$\Gamma(r) = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{A_i(r)}\frac{dN_i(r)}{dr} \rightarrow r^{D-3}$$

Estimates the average change in number of points within a spherical region of radius r, centered at an arbitrary point *i* within the space.

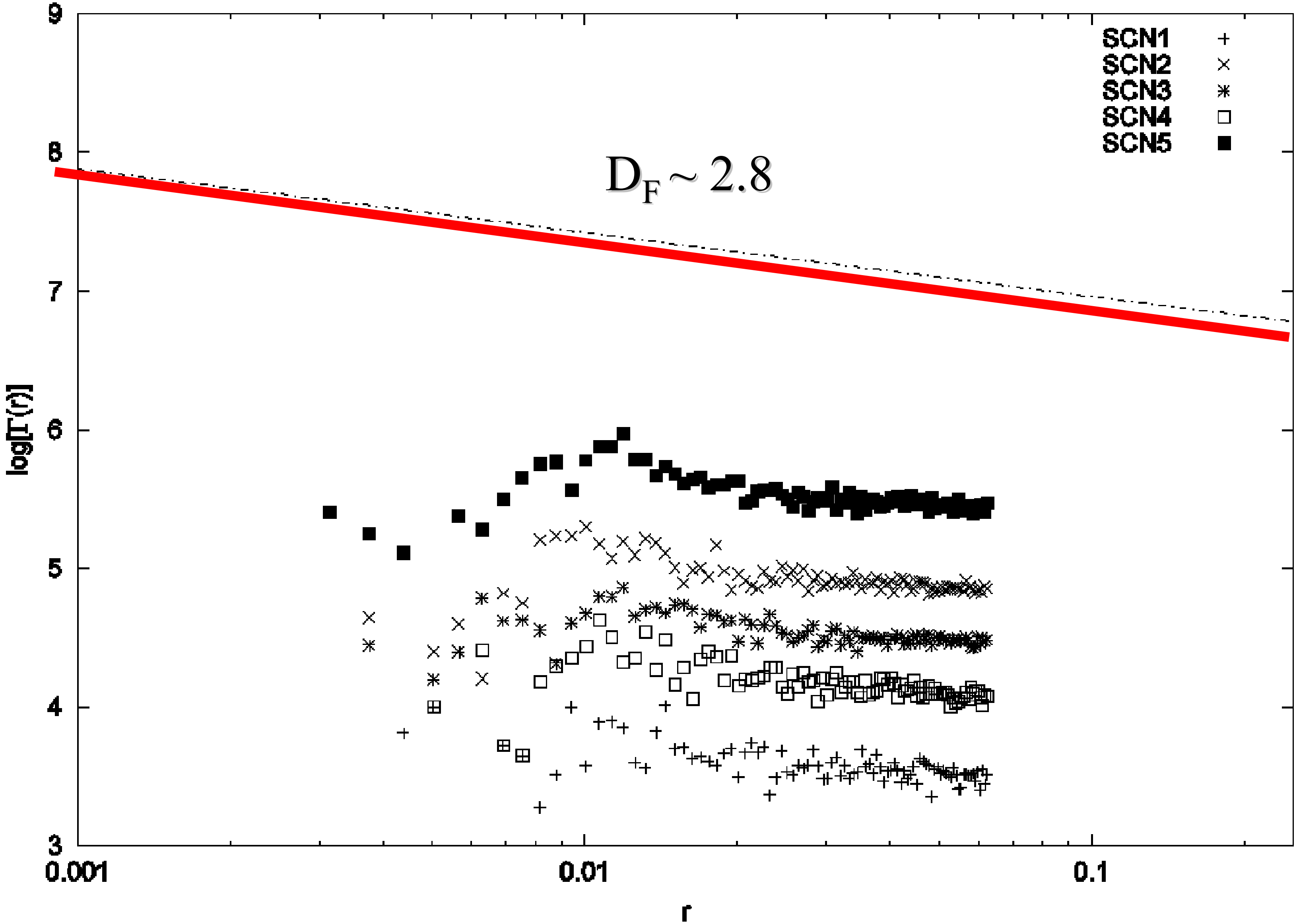
Flat curvature



Positive curvature



Negative curvature



So…..

- Can't explicitly determine geometry of Universe from fractal dimension
- No apaprent match to observation
- Is fractal dimension a useless statistic?
- $D_F \sim 2.5$ is roughly fractal dimension of 3-dimensional Appolonian packing
- What else can we learn from it?

Not Just Fractals….

Multifractals!

- Object whose structure cannot be described by a single scaling behavior
- Union of an infinite number of fractals
- How to measure?
  - Modify Box Counting to account for local "density" of pattern in each cell

Natural Multifractals

- The world isn't really "Euclidean"
- Many objects in Nature exhibit fractal-like structure; more details at higher magnifications
- *Statistical* self-similarity over a *finite* range of length scales
- Can study vegetation growth patterns, traffic flow (timeseries), topographical terrain, artwork

Multifractal Measure

- Density

$$p_i(d) = \frac{n_i(d)}{N_{tot}}$$

- Measure

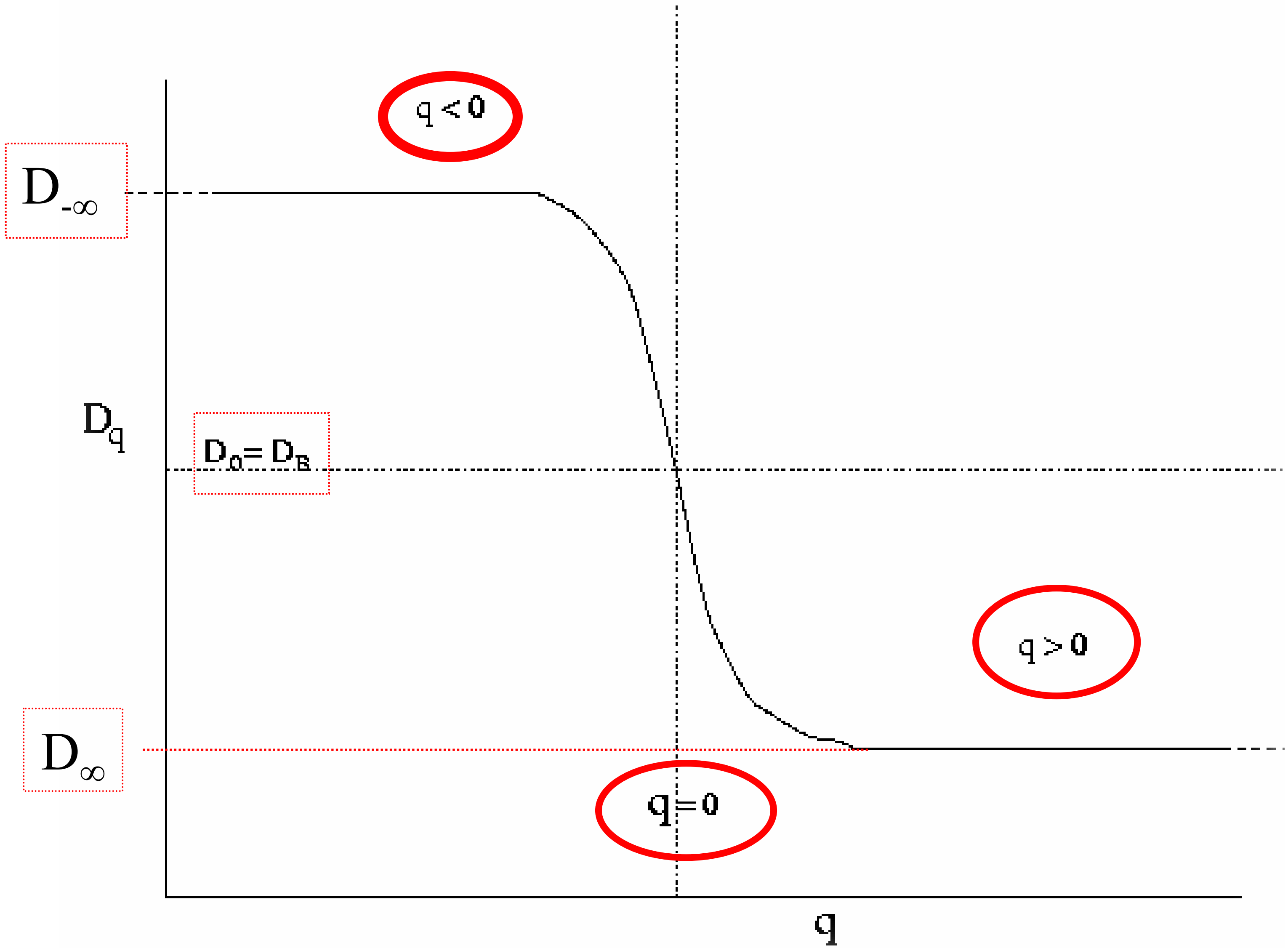
$$Z(q,d) = \sum_{i=1}^{N(d)} [p_i(d)]^q$$

- MF Dimension

$$\tau(q) = \frac{d \log[Z(q,d)]}{d \log[d]}$$

$$D_q = \frac{\tau(q)}{q-1}$$

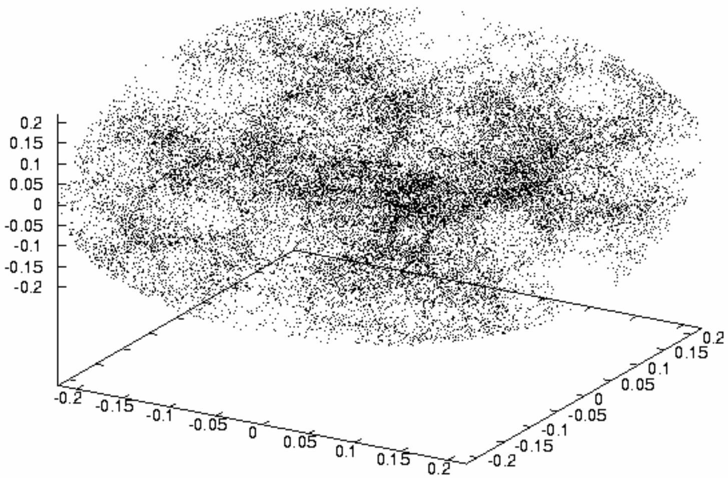
Multifractal Dimension Spectrum



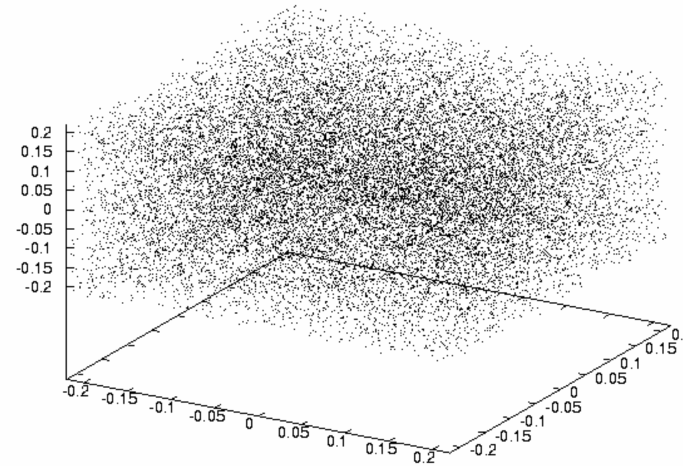
What do the D_q tell us?

- q is a “filter” or “magnifying glass”
- $q > 0$ highlights dense portions of the pattern
- $q < 0$ highlights sparse portions of the set
- $q \rightarrow \infty$ shows *strongest clustering regions*
- $q \rightarrow -\infty$ shows *least dense regions*

Swiss Cheese Model

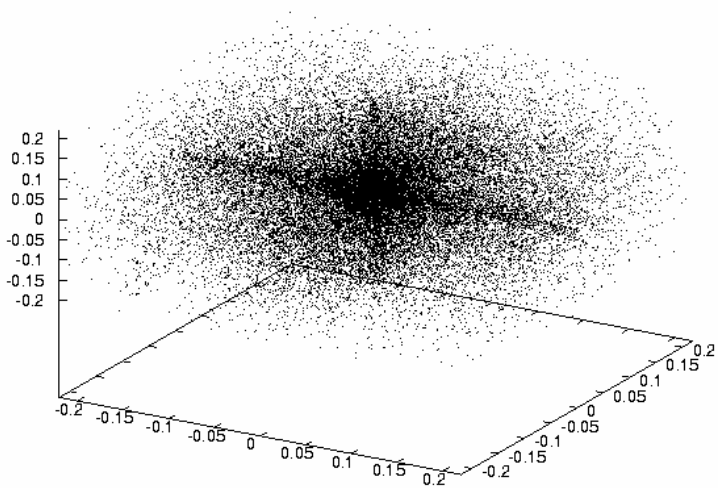


Random points

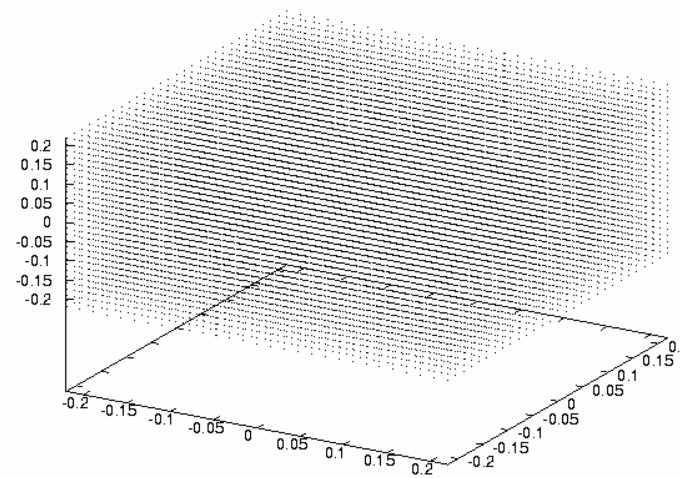


$D \approx 2.8$

$D = 3$

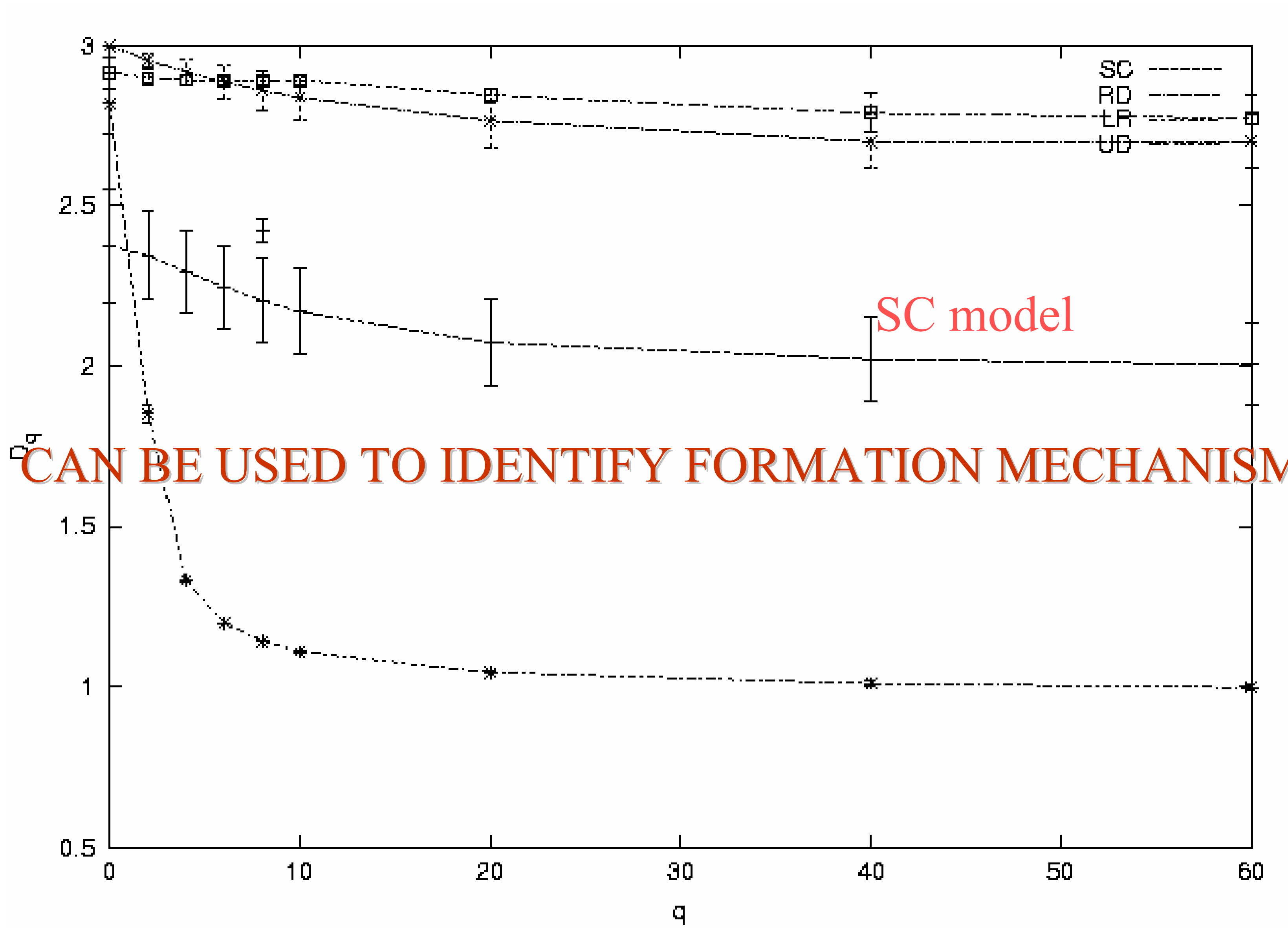


"Pseudo-linear"



Linear

Multifractal Spectra of Distributions



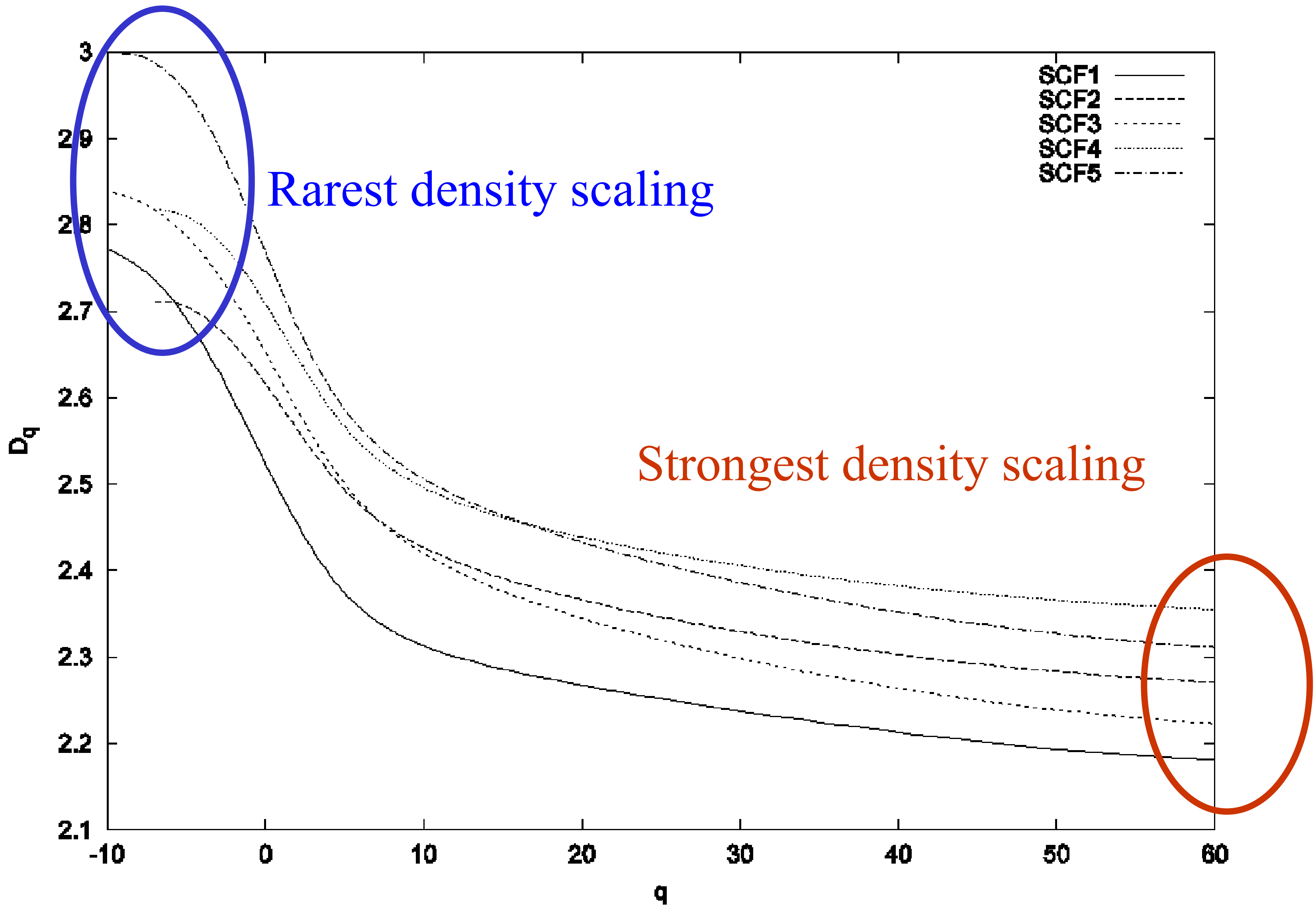
CAN BE USED TO IDENTIFY FORMATION MECHANISM

Geodesic Considerations: Density Reconstruction Function

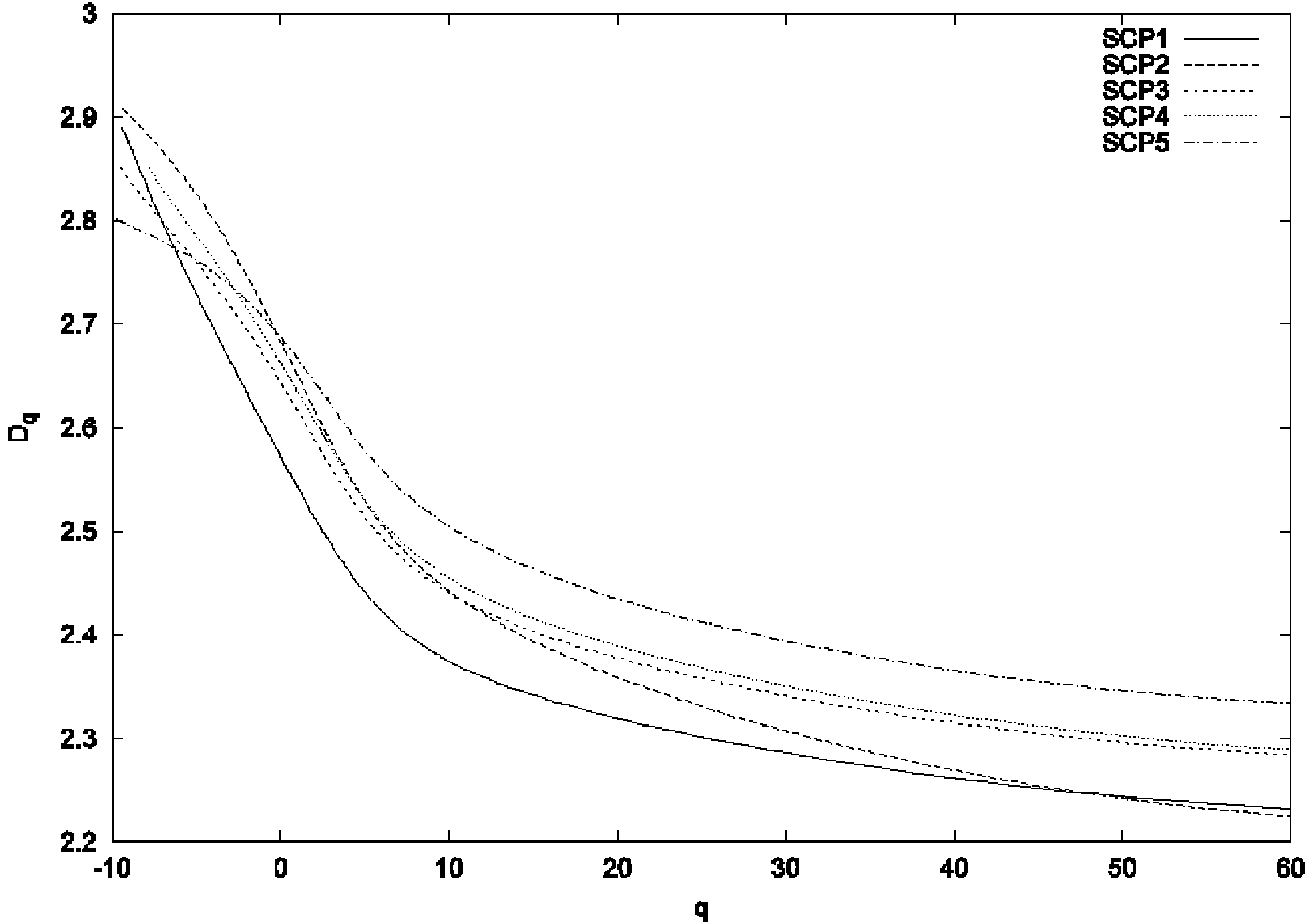
- As with regular fractal dimension, box counting to determine multifractal spectrum seems limiting
- Can use the *density reconstruction function* to determine multifractal moments from a "bottom-up" approach (as with conditional density)
- Evaluate along geodesic distance from randomly selected point in catalog

$$W(\tau, q) = \frac{1}{N}\sum_{i=1}^{N} r_i(p)^{-\tau} \rightarrow p^{1-q}$$

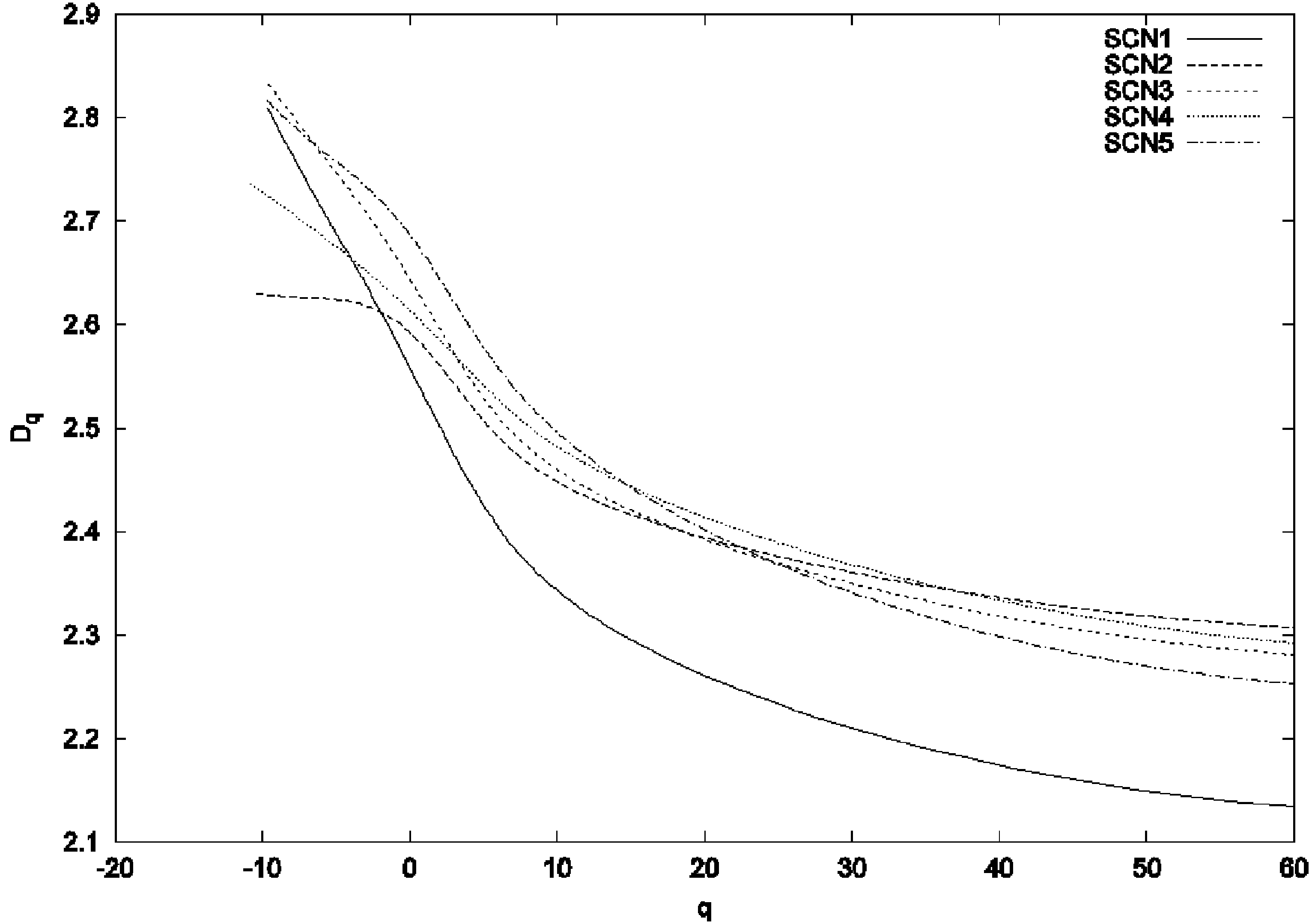
Density Reconstruction Multifractal Spectrum (Flat)



Density Reconstruction Multifractal Spectrum (Positive)



Density Reconstruction Multifractal Spectrum (Negative)



General Summary of Multifractal Results

	Swiss Cheese	Observation
$D_{-\infty}$	3	3
D_0	2.5-2.8	2
D_∞	2	1

- No explicit match between theory and experiment
  - *Different formation models*
- $D_\infty = 2$; Strongest clustering occurs on *surfaces* for SC
- Observation suggests local clustering is $D = 2$; strongest clustering is *filametary,* $D_\infty = 1$ [Filaments on sheets]

Even though they look the same, structure is different due to *construction paradigm*

Luminosity Basing Considerations

- Model isn't doomed!
- If we account for luminosity biasing considerations, then fractal analysis methods can yield seemingly lower dimensions (sample-size dependence??)
- Assume $M(L) \sim L^{\beta}$ $\beta \sim 1$, use mass cut-off roughly 0.01% that of largest in catalog
- SSRS data analysis suggests strong connection between statistical clustering behavior and luminosity, weighted toward brighter galaxies [Benoist *et al.,* 1996], and earlier evidence of a mismatch between galaxy correlation functions w/r to cluster richness [Bahcall and Soneira. 1983]

Current Projects and Future Directions

- Hybrid model of PSC with “N-body” clustering on surfaces?
- Fractality and curvature
  - How does curvature affect the notion of fractal structure? [Dyer and Mureika, in preparation]
  - Traditional fractal dimension seems rooted in Euclidean definitions
  - Hilbert’s Congruence Axioms on curved manifolds?
- Lacunarity analysis of clustering
  - Voids instead of clumps!
  - Can curvature be detected this way?

Acknowledgments

This work is supported by a Postdoctoral Fellowship From the Natural Sciences and Engineering Research Council of Canada. Thanks to Charles Dyer (University of Toronto) for continued collaboration, and to Pitzer College for additional Financial support.

jmureika@jsd.claremont.edu

J. R. Mureika and C. C. Dyer, "Multifractal Analysis of Packed Swiss Cheese Cosmologies" [in press, *Classical and Quantum Gravity*]