Relativistic theory of surficial Love numbers

Eric Poisson

Department of Physics, University of Guelph

APS April Meeting 2013, Denver
Newtonian tides 1

In Newtonian theory, the tidal environment of a body of mass $M$ and radius $R$ is described by the tidal quadrupole moment

$$\mathcal{E}_{ab} = -\partial_{ab}U_{\text{ext}}(0) \sim M_{\text{ext}}/r_{\text{orb}}^3$$

The tidal response is measured either in terms of the body’s quadrupole moment

$$Q_{ab} = -\frac{2}{3}k_2 R^5 \mathcal{E}_{ab}$$

$k_2 =$ gravitational Love number

or in terms of the surface deformation

$$\frac{\delta R}{R} = -\frac{1}{2} h_2 \frac{R^4}{M} \mathcal{E}_{ab} n^a n^b$$

$h_2 =$ surficial Love number

$$n^a = x^a/r = [\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta]$$
Equivalently, the surface deformation can be described in terms of the curvature perturbation

\[ \frac{\delta R}{R} = -2 h_2 \frac{R^3}{M} \mathcal{E}_{ab} n^a n^b \]

The gravitational Love number \( k_2 \) is useful to astronomers. The surficial Love number \( h_2 \) is useful to geologists.

For perfect fluids, they are related by

\[ h_2 = 1 + 2k_2 > 1 \]
In general relativity, the tidal environment is described by $E_{ab}$, $B_{ab}$, the electric and magnetic components of the asymptotic Weyl tensor.

The gravitational Love number $k_2$ was given a proper relativistic definition. [Damour and Nagar (2009); Binnington and Poisson (2009)]

It was computed for polytropes and realistic models of neutron stars. [Potnikov, Prakash, Lattimer (2010); Hinderer, Lackey, Lang, Read (2010)]

The ability to measure the tidal deformation of neutron stars through its impact on gravitational waves was quantified. [Flanagan and Hinderer (2008); Hinderer, Lackey, Lang, Read (2010); Damour, Nagar, Villain (2012)]

The surficial Love number $h_2$ has not yet received a proper relativistic treatment. [Coordinate-dependent definition: Damour and Lecian (2009)]
The boundary of a deformed body traces a timelike hypersurface in spacetime, defined by $p + \delta p = 0$.

This hypersurface is foliated by closed two-surfaces. In the regime of slow tides (orbital timescale long compared with internal dynamical timescale), there is a preferred (static) slicing for the foliation.

The two-surfaces have a well-defined intrinsic geometry, and the tidal deformation can be described by

$$\frac{\delta R}{R} = -2h_2 \frac{R^3}{M} \mathcal{E}_{ab} n^a n^b$$

This involves only $\mathcal{E}_{ab}$.

The relativistic surficial Love number $h_2$ is gauge-invariant in the regime of slow tides.
The notion of surface deformation extends to black holes.

The intrinsic geometry of a deformed event horizon is invariant under a reparametrization of the horizon’s null generators (gauge invariant). [Vega, Poisson, Massey (2011)]

For a Schwarzschild black hole,

\[
\frac{\delta R}{R} = -4M^2 \mathcal{E}_{ab} n^a n^b \implies h_2 = \frac{1}{4} \quad \text{[Damour and Lecian (2009)}]
\]

The surface deformation of a Schwarzschild black hole involves only \( \mathcal{E}_{ab} \).
Rotating black hole 1

The tidal deformation of a Kerr horizon can be calculated to all orders in $\chi = a/M = J/M^2$ and described in a gauge-invariant way through $\delta R$.

The result is a long mess. Because the tidal deformation is added to a significant rotational deformation, there is no easy interpretation in terms of Love numbers.

The answer simplifies in the slow-rotation limit. To first order in $\chi$, there is no rotational deformation, and

$$\frac{\delta R}{R} = -4M^2 \mathcal{E}_{ab} n_*^a n_*^b + \frac{20}{3} M^2 \chi \langle a B_{bc} \rangle n^a n^b n^c + O(\chi^2)$$

$$n_*^a = [\sin \theta \cos(\phi + \frac{7}{6} \chi), \sin \theta \sin(\phi + \frac{7}{6} \chi), \cos \theta]$$
The dragging of inertial frames produces an angular shift of $\frac{7}{6} \chi$.

[Hartle (1974)]

It is also couples the spin vector $\chi^a$ to the magnetic tidal field $B_{ab}$ to produce an octupole ($\ell = 3$) deformation.

These effects must also be present in the tidal deformation of material bodies, and their description involves the introduction of new Love numbers.

These can be computed on the basis of the standard theory of slowly-rotating stars.  [Hartle 1967, Hartle and Thorne 1968, Yagi and Yunez (2013)]