Tidal deformation and dynamics of compact bodies

Eric Poisson

Department of Physics, University of Guelph

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- Newtonian tides
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Goal and motivation

Goal
To describe the small deformations of a compact body created by tidal forces, and to explore their dynamical consequences.

Motivation
Tidal interactions are important in extreme mass-ratio inspirals: torquing of the large black hole leads to a gain of orbital angular momentum, with a significant impact on the orbital evolution (self-force effect). [Hughes (2001); Martel (2004); Yunes et al. (2010, 2011)]


The relativistic theory of tidal deformation and dynamics should be as complete as the Newtonian theory.
We consider a self-gravitating body ("the body") in a binary system with a companion body ("the companion").

The body has a mass $M$ and radius $R$. It is spherical in isolation.

The companion is far from the body: $r_{\text{orb}} \gg R$.

The tidal forces are weak and the deformation is small.

The orbital period is long compared with the time scale associated with hydrodynamical processes in the body; the tides are slow.

We work in the body’s moving frame, with its origin at the centre-of-mass.

We focus attention on a neighbourhood that does not extend too far beyond the body; it does not include the companion.
Gravitational potential

**External potential**

\[ U_{ext}(t, \mathbf{x}) = U_{ext}(t, \mathbf{0}) + g_a(t)x^a - \frac{1}{2}\mathcal{E}_{ab}(t)x^a x^b + \cdots \]

\[ g_a(t) = \frac{\partial}{\partial x^a} U_{ext}(t, \mathbf{0}) = \text{CM acceleration} \]

\[ \mathcal{E}_{ab}(t) = -\frac{\partial^2}{\partial x^a \partial x^b} U_{ext}(t, \mathbf{0}) = \text{tidal tensor} \]

**Body potential**

\[ U_{body}(t, \mathbf{x}) = \frac{GM}{r} + \frac{3}{2}GQ_{ab}(t)\frac{x^a x^b}{r^5} + \cdots \]

\[ Q_{ab}(t) = \int \rho \left( x_a x_b - \frac{1}{3}r^2 \delta_{ab} \right) d^3x = \text{quadrupole moment} \]
The quadrupole moment $Q_{ab}$ measures the body’s deformation.

The central problem of tidal theory is to relate $Q_{ab}$ to the tidal tensor $E_{ab}$; this requires solving the equations of hydrostatic equilibrium for the perturbed configuration.

$$GQ_{ab} = -\frac{2}{3} k_2 R^5 E_{ab}$$

$k_2$ = gravitational Love number

$$U = \frac{GM}{r} - \frac{1}{2} \left[ 1 + 2k_2 (R/r)^5 \right] E_{ab} x^a x^b$$

The Love number $k_2$ encodes the details of internal structure.
The preceding results don’t account for dissipation within the body.

Dissipation, such as created by viscosity, produces a short delay in the body’s response to the tidal forces.

\[
GQ_{ab}(t) \simeq -\frac{2}{3}k_2 R^5 \mathcal{E}_{ab}(t - \tau)
\]

\[
\tau = \text{viscous delay}
\]

The viscous delay depends on the precise mechanism responsible for dissipation and the details of internal structure; it is typically much shorter than the orbital period.
Lag and lead

The delay creates a misalignment between the direction of the tidal bulge and the direction of the companion.

**When** $\Omega_{\text{body}} < \Omega_{\text{orbit}}$

Orbital motion is **faster** than body’s intrinsic rotation.

Tidal bulge **lags** behind the companion.

Tidal torquing **increases** the body’s angular momentum.

**When** $\Omega_{\text{body}} > \Omega_{\text{orbit}}$

Orbital motion is **slower** than body’s intrinsic rotation.

Tidal bulge **leads** in front of the companion.

Tidal torquing **decreases** the body’s angular momentum.
Tidal dynamics

Dissipation permits an exchange of angular momentum between the body and the orbit.

### Tidal torquing

\[
\frac{dJ}{dt} = 6\left(k_2 \tau\right) \frac{GM_{\text{com}}^2 R^5}{r_{\text{orb}}^6} \left(\Omega_{\text{orbit}} - \Omega_{\text{body}}\right)
\]

This eventually leads to tidal locking: \(\Omega_{\text{body}} = \Omega_{\text{orbit}}\).

Dissipation is accompanied by the generation of heat.

### Tidal heating

\[
\frac{dQ}{dt} = 6\left(k_2 \tau\right) \frac{GM_{\text{com}}^2 R^5}{r_{\text{orb}}^6} \left(\Omega_{\text{orbit}} - \Omega_{\text{body}}\right)^2
\]
Relativistic theory

A relativistic theory of tidal deformation and dynamics must provide meaningful generalizations of the Newtonian quantities

$$\mathcal{E}_{ab} = -\partial_{ab}U_{\text{ext}} = \text{tidal tensor}$$

$$k_2 = \text{gravitational Love number}$$

$$U = \frac{GM}{r} - \frac{1}{2} \left[ 1 + 2k_2 \left( \frac{R}{r} \right)^5 \right] \mathcal{E}_{ab} x^a x^b$$

A tidal tensor $\mathcal{E}_{ab}$ can be defined in terms of the Weyl tensor evaluated at a distance $r$ such that $R \ll r \ll r_{\text{orb}}$.

The deformation of the gravitational field from a spherical configuration can be described as a perturbative expansion.

The construction provides a relativistic, gauge-invariant definition for $k_2$. [Damour & Nagar (2009); Binnington & Poisson (2009)]
Metric of a tidally deformed body

\[ g_{tt} = -1 + 2U_{\text{eff}}/c^2 \]

\[ U_{\text{eff}} = \frac{M}{r} - \frac{1}{2} \left[ A(r) + 2k_2 (R/r)^5 B(r) \right] \varepsilon_{ab} x^a x^b \]

\[ A(r) = \text{simple polynomial in } 2M/r \]

\[ B(r) = \text{complicated function of } 2M/r \]

The relativistic Love number \( k_2 \) is calculated by solving the equations of hydrostatic equilibrium for the body’s perturbed configuration.

The results depend on the body’s equation of state \( p(\rho) \) and its compactness \( M/R \).
Relativistic Love numbers

\[ k_2 \text{ for polytropes: } p = K \rho^{1+1/n} \]

[Damour & Nagar (2009)]

For black holes, \( k_2 = 0 \) [Binnington & Poisson (2009)]
Applications require the computation of the tidal tensor $\mathcal{E}_{ab}$.

This involves inserting the black hole in a larger spacetime that contains additional bodies, and matching the metrics.

[Taylor & Poisson (2008); Johnson-McDaniel et al (2009)]

For a binary system in post-Newtonian circular orbit,

$$\mathcal{E}_{12} = -\frac{3M_{\text{com}}}{2r_{\text{orb}}^3} \left[ 1 - \frac{3M + 4M_{\text{com}}}{2(M + M_{\text{com}})} v^2 + O(v^4) \right] \sin(2\omega t)$$

$$\omega = \Omega \left[ 1 - \frac{M M_{\text{com}}}{(M + M_{\text{com}})^2} v^2 + O(v^4) \right]$$

The angular frequency $\omega$ differs from the orbital angular velocity $\Omega$: the transformation from the global inertial frame to the black hole’s moving frame involves time dilation and rotation.
Dissipation

There is currently no relativistic theory of tidal interactions that includes dissipation, except for black holes. [On my todo list]

The event horizon of a black hole naturally provides a dissipation mechanism: the black hole absorbs whatever crosses its boundary.

The horizon’s null generators can be thought of as the streamlines of an effective fluid. [Membrane paradigm]

This fluid possesses an effective viscosity.

The absorbing properties of the event horizon lead to the tidal torquing and heating of the black hole.
Tidal dynamics of black holes (1)

Black-hole dynamics

\[
\frac{dJ}{dt} = \frac{32}{5} \frac{M_{\text{com}}^2 M^6}{r_{\text{orb}}^6} (\Omega_{\text{orbit}} - \Omega_{\text{hole}})
\]

\[
\frac{dQ}{dt} = \frac{32}{5} \frac{M_{\text{com}}^2 M^6}{r_{\text{orb}}^6} (\Omega_{\text{orbit}} - \Omega_{\text{hole}})^2
\]

Newtonian dynamics

\[
\frac{dJ}{dt} = 6(k_2 \tau) \frac{G M_{\text{com}}^2 R^5}{r_{\text{orb}}^6} (\Omega_{\text{orbit}} - \Omega_{\text{body}})
\]

\[
\frac{dQ}{dt} = 6(k_2 \tau) \frac{G M_{\text{com}}^2 R^5}{r_{\text{orb}}^6} (\Omega_{\text{orbit}} - \Omega_{\text{body}})^2
\]
Tidal dynamics of black holes (2)

Comparison produces \( R \propto GM/c^2 \) and \( (k_2 \tau) \propto GM/c^3 \).

More precisely

\[
(k_2 \tau) R^5 = \frac{16}{15} \frac{(GM)^6}{c^{13}}
\]

The quantity \((k_2 \tau)\) is assigned a nonzero value in spite of the fact that \( k_2 = 0 \) for a black hole!

The tidal dynamics of black holes is in near-quantitative agreement with the tidal dynamics of viscous Newtonian bodies.

Relativistic corrections: [Taylor & Poisson (2008); Chatziioannou et al (2013)]

\[
\frac{dJ}{dt} = -\frac{32}{5} \frac{M_{\text{com}}^2 M^6}{r_{\text{orb}}^6} \Omega_{\text{hole}}
\quad \times \frac{1}{2} (1 + \sqrt{1 - \chi^2}) \left\{ 1 + 3\chi^2 - \frac{1}{4} \left[ 12 + 51\chi^2 - \frac{(4 + 12\chi^2)M}{M + M_{\text{com}}} \right] v^2 + O(v^3) \right\}
\]
Conclusion

The Newtonian theory of tidal deformations and dynamics is undergoing a generalization to relativistic gravity.

A meaningful description of the tidal deformation of a relativistic body has been achieved; the gravitational Love numbers have been ported to general relativity.

The tidal dynamics of black holes is well developed; it displays a remarkable similarity with the Newtonian theory of viscous bodies.

The tidal dynamics of material bodies remains to be developed; this requires the incorporation of viscosity.

**Recent work:** Surficial love numbers [Landry & Poisson (2014)]

**Current work:** Slowly rotating bodies.

**Future work:** Higher dimensions.
Exploring approximate solutions to general relativity and their consequences, this textbook offers a unique presentation of Einstein's theory by developing powerful methods that can be applied to astrophysical systems.

Beginning with a uniquely thorough treatment of Newtonian gravity, the book develops post-Newtonian and post-Minkowskian approximation methods to obtain weak-field solutions to the Einstein field equations. It explores the motion of self-gravitating bodies, the physics of gravitational waves, and the impact of radiative losses on gravitating systems. The book concludes with a brief overview of alternative theories of gravity.

Ideal for graduate courses on general relativity and relativistic astrophysics, the book examines real-life applications, such as planetary motion around the Sun, the timing of binary pulsars, and gravitational waves emitted by binary black holes. Text boxes explore related topics and provide historical context, and over 100 exercises present interesting and challenging tests of the material covered in the main text.

Eric Poisson is Professor of Physics at University of Guelph. He is a Fellow of the American Physical Society and serves on the Editorial Boards of *Physical Review Letters* and *Classical and Quantum Gravity*.

Clifford M. Will is Distinguished Professor of Physics at the University of Florida and J. S. McDonnell Professor Emeritus at Washington University in St. Louis. He is a member of the US National Academy of Sciences, and Editor-in-Chief of *Classical and Quantum Gravity*. He is well known for his ability to bring science to broad audiences.

Endorsements to follow.

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