Travelling and Standing Waves

Many biological phenomena are cyclic.

- Heartbeats
- Circadian rhythms,
- Estrus cycles
- Many more e.g.’s

Such events are best described as waves.

Therefore the study of waves is a major component of this biophysics course.

**Sound Waves:**
(Acoustics, vibrating strings)

**Electromagnetic Waves:**
(Light, X-rays, Infra-red)

**Quantum Mechanical Waves:**
(motion of elementary particles/waves)

Study guide 3-4

Study guide 4-6

Study guide 2
**MATERIAL TO READ**

**Web:**
http://www.physics.uoguelph.ca/biophysics/
www.physics.uoguelph.ca/tutorials/GOF/
... /tutorials/trig/trigonom.html
... /tutorials/shm/Q.shm.html

**Text:** chapter 1, Vibrations and waves

**Handbook:** study guide 1

**For 1st quiz, you should know:**

1. How to evaluate $\sin(x)$, $\cos(x)$, $\sin^{-1}(x)$, $\cos^{-1}(x)$
2. Equations for travelling and standing waves
3. How to graph a wave given the equation
4. Relations between period ($T$), frequency ($f$ or $\nu$) and wavelength ($\lambda$)

<table>
<thead>
<tr>
<th>Type of wave</th>
<th>Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sinusoidal</td>
<td>$y = A\sin(x)$</td>
</tr>
<tr>
<td>Square</td>
<td>$y = A$ for $0 &lt; x &lt; \frac{\lambda}{2}$, $= -A$ for $\frac{\lambda}{2} \leq x \leq \lambda$</td>
</tr>
<tr>
<td>Ramp</td>
<td>$y = Cx$, for $-\frac{\lambda}{2} \leq x \leq \frac{\lambda}{2}$</td>
</tr>
</tbody>
</table>

Periodic functions can be described as a sum of sinusoidal waves. Therefore we need trigonometry
Trigonometry Review

Trig. Functions defined on a circle...

Point P moves around a circle of radius $r$

$$\theta = \frac{s}{r}$$

dimensionless unit: “radians”

See Web tutorial to review dimensions

Common unit is “degrees”:

Full circle = 360°

For full circle, $s =$ circumference $= 2\pi r$

$$\therefore \frac{s}{r} = 2\pi = 360^\circ$$

$$\Rightarrow 1 \text{ radian} = \frac{360^\circ}{(2\pi)} = 57.3^\circ.$$
Definitions:

<table>
<thead>
<tr>
<th>Function</th>
<th>Formula</th>
<th>Description</th>
<th>Abbreviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin(θ)</td>
<td>y/r</td>
<td>Opposite/hypotenuse</td>
<td>SOH</td>
</tr>
<tr>
<td>cos(θ)</td>
<td>x/r</td>
<td>Adjacent/hypotenuse</td>
<td>CAH</td>
</tr>
<tr>
<td>tan(θ)</td>
<td>y/x</td>
<td>Opposite/adjacent</td>
<td>TOA</td>
</tr>
</tbody>
</table>

Be careful calculating these functions using calculator: “DEG” or “RAD” mode on calculator indicates units.

Examples:  
\[
\sin(40.0^\circ) = 0.642 \\
\cos(2.5) = -0.801
\]

As P moves around circles \( \sin(\theta) \) and \( \cos(\theta) \) vary...
NOTES:

1. Positive angles are measured counterclockwise from positive x axis

2. Both $\sin \theta$ and $\cos \theta$ repeat once every time P rotates around circle ($2\pi$ radians)
   
   i.e. for any integer $n$,
   
   $\sin(\theta + 2\pi n) = \sin(\theta)$
   
   $\cos(\theta + 2\pi n) = \cos(\theta)$

3. Also:

   $\sin(-\theta) = -\sin(\theta)$ ODD
   
   $\cos(-\theta) = \cos(\theta)$ EVEN
**Small Angle Approximation:**

For $\theta < \pi/20$ radians $\cong 10^\circ$:  
$\theta$ MUST BE IN RADIANS  

- $\sin \theta \cong \theta$  
- $\cos \theta \cong 1$  
- $\tan \theta = \sin \theta / \cos \theta \cong \theta$

Not used until Study Guide 4.

**Inverse Trig. Functions: Inverse Sine**

$\sin \theta = f$ (given $f$)  

What is $\theta$?

Imagine there is an inverse function for sin. Let’s say it is called $\sin^{-1}$. Let’s apply this function to both sides of the equation.

$\sin^{-1}(\sin(\theta)) = \sin^{-1}(f)$  
$\Rightarrow \theta = \sin^{-1}(f)$

**LOOK!**  
**THERE IS SIN-1 ON YOUR CALCULATOR**
• sin-1 also called “arcsin(f)”
• similar functions for cos-1(f) and tan-1(f).

**Example:**

Calc. setting

\[
\sin^{-1}(0.683) = 0.752 \quad (R: \text{ radians})
\]

\[
= 43.1^\circ \quad (D: \text{ degrees})
\]

Due to the periodic nature of sin, cos and tan there are actually an infinite set of angles that would produce the same value of each function...

All 4 values of \( \theta \) have same \( f \); only 2 per revolution.
Therefore $\sin^{-1}(f)$ is defined on interval $0 \leq f \leq 2\pi$.
The two values are: $\theta$ \textbf{AND} $(\pi - \theta) = (180^\circ - \theta)$.

The reason there are two values per revolution can be understood with the aid of a diagram.

![Diagram showing two angles $\theta_1$ and $\theta_2$ with corresponding $y_1$ and $y_2$ values.]

$y_1 = \sin(\theta_1) = \sin(\theta_2) = y_2$

**Example:** If $\sin \theta = 0.45$, $\theta = ?$

Calculator returns $\theta = \sin^{-1}(0.45) = 26.7^\circ$

Another value is: $180^\circ - 26.7^\circ = 153^\circ$

**THIS ALSO WORKS IF $\theta$ IS NEGATIVE**

Flip the above diagram around to see it.

Example $\theta = \sin^{-1}(-0.48)$...
**Inverse Cosine:**

**Example:** \( \theta = \cos^{-1}(0.45) = 63.3^\circ \)

What is other value?

\( x_1 = \cos(\theta_1) = \cos(\theta_2) = x_2 \)
From the diagram we see that the other value for $\theta$ is, $\theta_2 = 360^\circ - \theta_1 = 296.7^\circ = -\theta_1$

NB!

If we flip the above diagram, we see that this also works for $\theta < 0$.

Therefore $\cos^{-1}(f)$ is also defined on interval $0 \leq f \leq 2\pi$. Its 2 values are: $\theta$ AND $(2\pi - \theta) = (360^\circ - \theta)$.

**Inverse tangent:**

![Graph of tangent function with values $\theta_1$ and $\theta_2$]

**Example:** $\theta = \tan^{-1}(1.0) = 45.0^\circ$

What is other value?
From the diagram we see that the other value for \( \theta \) is, \( \theta_2 = \theta_1 - 180^\circ = -135.0^\circ = 225.0^\circ \).

If we flip the above diagram, we see that this also works for \( \theta < 0 \).

Therefore \( \tan^{-1}(f) \) is also defined on interval \( 0 \leq f \leq 2\pi \). Its 2 values are: \( \theta \text{ AND } (\theta - \pi) = (\theta-180^\circ) \).

See Study Guide for more examples to practice on...
Oscillations and Travelling Waves

e.g. hanging weight on spring

Spring stretched to initial position, $y_0$, and released. The mass oscillates between $y_0$ and $-y_0$ in "Simple Harmonic Motion" or SHM.

Equation of motion:
\[ \Sigma F = ma \Rightarrow -ky = ma_y \quad \text{(Ideal Spring Approx.)} \]

\[ \Rightarrow -(k/m)y = (d^2y/dt^2) \quad \text{Diff. Equation} \]

Let's try:
\[ y(t) = y_0 \sin(\omega t) \Rightarrow \frac{dy}{dt} = \omega y_0 \cos(\omega t) \]

where \( \omega = \frac{2\pi}{T} \) = angular frequency
and \( T \) = period = time for one cycle.

NB: \( y(t) = y_0 \sin(2\pi(t/T)) \) repeats every \( T \)

Argument MUST be in radians

\[ \Rightarrow \frac{d^2y}{dt^2} = -\omega^2 y_0 \sin(\omega t) \]

\[ \Rightarrow \frac{d^2y}{dt^2} = -\omega^2 y(t) \]

\[ \therefore y(t) = y_0 \sin(\omega t) \] is solution to diff. eq. if \( \omega^2 = k/m. \)

- This is why SHM is sinusoidal.
- Other e.g.'s: sound, light, water surface ripples.
- Looks like:

![Diagram](attachment:image.png)
Other useful definitions:

Frequency = \( f \), or \( \nu \) = number of cycles or oscillations per unit time (usually seconds).

E.g.: Let, \( T = 0.5 \text{ s} \) (period)

in one second 2 oscillations occur

that is: \( \nu = 2 \text{ cycles/second} \)

\[ \nu = \frac{1}{T} = \frac{\omega}{2\pi} \]

Dimensions of \( \nu \): (time\(^{-1}\))

Unit: s\(^{-1}\)

or more explicitly, "cycles per second" \( \equiv \) Hertz

So we can write equivalent representations:

\[ y(t) = y_0 \sin(\omega t) \]

\[ y(t) = y_0 \sin\left(\frac{2\pi t}{T}\right) \]

\[ y(t) = y_0 \sin(2\pi \nu t) \]
**Travelling Waves**

e.g.: ripples on surface of water. Displacement is time and position dependent. Let's take “snapshots” of y vs. x at three successive times:

(earliest, middle, latest)

peaks moving towards right

each point moves ONLY vertically

y as a function of x at any fixed time is sinusoidal

wave repeats itself after distance, $\lambda =$ wavelength.
y as a function of t at any fixed position is also sinusoidal

In one period $T$, wave moves forward 1 wavelength.

speed of Wave = Distance/time = $\lambda / T = \lambda v$

$\therefore v = \lambda v$
governs all wave motion (water ripples, sound, light...)

for velocity must also provide direction along +ve or -ve x.

**Equations for Travelling Wave (Function of t, x)**

$y = y_0 \sin(\omega t \pm kx)$ “+” or “-” according to direction

where $k = 2\pi / \lambda = $ wave number

$y = y_0 \sin(2\pi t / T \pm 2\pi x / \lambda)$

$y = y_0 \sin(2\pi vt \pm kx)$

$y = y_0 \sin(k(v t \pm x))$ since $kv = (2\pi / \lambda)v = 2\pi v = \omega$
If we look at a particular position, equation reduces: e.g. $x = 0 \Rightarrow y = y_0 \sin(2\pi t/T)$.

Alternatively can take snapshots at particular time, equation reduces again: $y = y_0 \sin(-2\pi x/\lambda)$ (we have assumed $+ve$ $x$ velocity for wave)

Wave direction? Crest of wave $\Rightarrow y = y_0$

$y = y_0 \sin(2\pi t/T \pm 2\pi x/\lambda)$

$\Rightarrow 1 = \sin(2\pi t/T \pm 2\pi x/\lambda)$

$\Rightarrow n\pi + \pi/2 = (2\pi t/T \pm 2\pi x/\lambda)$ (for $n = 0, 1, 2, \ldots$)

Let's use $n = 0$ case and solve for $x$ (position of crest) vs. $t$. 

$\pi/2 - 2\pi t/T = \pm 2\pi x/\lambda$
\[ \pm x = \frac{\lambda}{4} - \frac{\lambda t}{T} \]

since \[ \frac{\lambda}{T} = \frac{\lambda v}{v} = v \]

\[ \pm x = \frac{\lambda}{4} - vt \]

This separates into two equations:

Upper sign
\[ x = \frac{\lambda}{4} - vt \]

Lower sign
\[ -x = \frac{\lambda}{4} - vt \]

\[ \Rightarrow \quad x = vt - \frac{\lambda}{4} \]

-ve velocity +ve velocity

Therefore, in the equation:

\[ y = y_0 \sin(2\pi t/T \pm 2\pi x/\lambda) \]

The upper sign corresponds to wave crest velocities in the negative \( x \) direction and the lower sign to those in the positive \( x \) direction.
**Sample Problem** example 1-5 from the text

A wave moves along a string in the +x direction with a speed of 8.0 m/s, a frequency of 4.0 Hz and amplitude of 0.050 m. What are the...

(a) wavelength? (b) wave number?
(c) period? (d) Angular frequency?
(e) Determine an equation for this wave (and sketch it for t = 0 and t= 1/16 s)
(f) What is value of y for the point x = ¾ m a t time t = 1/8 s?

**Solution**

(a) \( v = \nu \lambda \Rightarrow \nu / \nu = \lambda = 8.0 / 4.0 = 2.0 \text{ m} \)

(b) \( k = 2\pi / \lambda = 2\pi / 2 = \pi \text{ m}^{-1} \)

(c) \( T = 1 / \nu = 1 / 4.0 = 0.25 \text{ s} \)

(d) \( \omega = 2\pi \nu = 8\pi \text{ s}^{-1} \)

(e) \( y = y_0 \sin(\omega t \pm kx) = 0.05 \sin(8\pi t - \pi x) \) -ve sign given by fact that wave travels in +ve x direction (see previous page. For \( t = 0 \), \( y = 0.05 \sin(- \pi x) = - 0.05 \sin(\pi x) : \)
for \( t = 1/16 \) s,
\[
y = 0.05 \sin(\pi/2 - \pi x) = -0.05 \sin(\pi(x - \frac{1}{2}))
\]
\[
\Rightarrow y = -0.05 \sin(\pi x'), \text{ same as above except } x' = x - \frac{1}{2}.
\]

redraw horizontal axis using \( x' + \frac{1}{2} = x \).

(f) \( y = 0.05 \sin(8\pi t - \pi x) \), for \( t = 1/8 \) s and \( x = \frac{3}{4} \) m.
\[
y = 0.05 \sin(\pi - 3\pi/4) = 0.05 \sin(\pi/4) = 0.035 \text{ m}
\]
**Sample Problem**  Self-test III, #1 c

Sketch a graph for $y = 4 \sin(3\pi t - 6\pi x)$ at $t = 0.900$ s.

\[
y = 4 \sin(3\pi(0.900) - 6\pi x) = 4 \sin(2.7\pi - 6\pi x) = -4 \sin(6\pi x - 2.7\pi) = -4 \sin(6\pi(x - (2.7/6)\pi))
\]

\[
y = -4 \sin(6\pi(x - 0.45\pi)) = -4 \sin(6\pi x')
\]

\[
x' = x - 0.45
\]

\[
k = 6\pi = 2\pi/\lambda \Rightarrow \lambda = 2\pi/6\pi = 1/3 \text{ m}
\]

Redraw using $x = x' + 0.45$

\[
y(m) \quad 1/6 + 0.45 \quad 1/3 + 0.45
\]

\[
= 0.62 \text{ m} \quad = 0.78 \text{ m}
\]

\[
x = 0.45 \text{ m}
\]
**Sample Problem**  
**Self-test III, #1 d**

Sketch \( y = 4 \sin(3\pi t - 6\pi x) \) at \( x = 0.5 \text{ m} \) (modified from \( x = 0.4 \text{ m} \)).

Since \( x \) is fixed \( \Rightarrow \) plot \( y \) vs. \( t \)

\[
\Rightarrow \quad y = 4 \sin(3\pi t - 6\pi(0.5)) = 4 \sin(3\pi t - 3\pi)
\]

\[
\Rightarrow \quad y = 4 \sin(3\pi(t - 1)) = 4 \sin(3\pi t')
\]

\( t' = t - 1.0 \)

\( \omega = 3\pi = 2\pi/T \quad \Rightarrow \quad T = 2\pi/3\pi = 2/3 \text{ s} \)

---

Redraw using \( t = t' + 1.0 \)

---

\( x = 1.0 \text{ m} \)
Standing Waves

Wave crests do not move

\[ \text{e.g.: Vibrating string, sound in organ pipes} \]

Consider first:

Reflection of travelling wave from surface

- Incident wave
- Wave inverts upon reflection
- Reflected wave
A standing wave is the sum of two travelling waves with opposing velocities.

Incident wave (towards \(-x\)) \[ y_1 = y_0 \sin(\omega t + kx) \]

Reflected wave (towards \(+x\)) \[ y_2 = -y_0 \sin(\omega t - kx) \]

**Inversion of wave**

**Superposition principle:** Resultant wave is the sum of the incident and reflected wave.

\[ y_{\text{tot}} = y_1 + y_2 = y_0[\sin(\omega t + kx) - \sin(\omega t - kx)] \]

Use trig. identity: \( \sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta \)

\[ y_{\text{tot}} = y = y_0[\sin(\omega t)\cos(kx) + \cos(\omega t)\sin(kx) - \sin(\omega t)\cos(kx) + \cos(\omega t)\sin(kx)] \]

\[ y = 2y_0[\cos(\omega t)\sin(kx)] \]

**Equation for a standing wave**

Product of a time dependent term, \(\cos(\omega t)\), and a position dependent term, \(\sin(kx)\).
$y$ can be expressed as: $y = a(t)\sin(kx)$, where $a(t) = 2y_0\cos(\omega t)$

Sketch of standing wave, $y$ vs. $x$ at different times

At all times, node positions ($y = 0$) are at same $x$

At all times, antinode positions are at same $x$
Sample Problem
An incident wave \( y_1 = 3\sin(2t - 3\pi x) \) and
a reflected wave \( y_2 = -3\sin(2t + 3\pi x) \)
produce a standing wave (\( x, y, t \) in SI units).

a) Write the equation of the standing wave.
Using the summation identity as shown previously we can write the following:

**General Prescription:**
- Incident wave: \( y_1 = y_0\sin(\omega t - kx) \)
- Reflected wave: \( y_2 = -y_0\sin(\omega t + kx) \)

\[ \downarrow \]
- Standing wave: \( y = -2y_0\cos \omega t \sin kx \)

In this case we obtain: \( y = -6\cos 2t \sin 3\pi x \)

b) What are the... (i) amplitude
standing wave's... (ii) period...
    (iii) and wavelength?

(i) amplitude = \( 2y_0 = 6 \text{ m} \).
(ii) \( \omega = 2 \text{ s}^{-1} = 2\pi/T \Rightarrow T = \pi \text{ s} \)
(iii) \( k = 3\pi \text{ m}^{-1} = 2\pi/\lambda \Rightarrow \lambda = 2/3 \text{ m} \)
c) Sketch the standing wave at times $t = 0 \text{ s}$, $t = T/4$ and $t = T/2$.

At $t = 0$

$y = -6\cos 2t \sin 3\pi x \Rightarrow y = -6\cos(2(0)) \sin 3\pi x$

$= -6\sin 3\pi x$

At $t = T/4 = \pi/4 \text{ s}$

$y = -6\cos 2t \sin 3\pi x \Rightarrow y = -6\cos(2\pi/4) \sin 3\pi x$

$y = 0$ for all $x$

At $t = T/2 = \pi/2 \text{ s}$

$y = -6\cos 2t \sin 3\pi x \Rightarrow y = -6\cos(2\pi/2) \sin 3\pi x$

$y = 6 \sin 3\pi x$ (opp. of $t = 0$ graph)
Physics diagram aids