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Calculation of thermal noise in atomic force microscopy

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Abstract. Thermal fluctuations of the cantilever are a fundamental source of noise in atomic force microscopy. We calculated thermal noise using the equipartition theorem and considering all possible vibration modes of the cantilever. The measurable amplitude of thermal noise depends on the temperature, the spring constant K of the cantilever and on the method by which the cantilever deflection is detected. If the deflection is measured directly, e.g. with an interferometer or a scanning tunneling microscope, the thermal noise of a cantilever with a free end can be calculated from $\sqrt{kT/K}$. If the end of the cantilever is supported by a hard surface no thermal fluctuations of the deflection are possible. If the optical lever technique is applied to measure the deflection, the thermal noise of a cantilever with a free end is $\sqrt{4kT/3K}$. When the cantilever is supported thermal noise decreases to $\sqrt{kT/3K}$, but it does not vanish.

1. Introduction

The atomic force microscope (AFM), invented by Binnig, Quate and Gerber [1], has become an important tool to image the topography of surfaces with atomic resolution [2, 3, 4]. In the AFM the sample is scanned by a tip which is mounted on a cantilever spring. While scanning the deflection of the cantilever is detected. A topographic image of the sample is obtained by plotting the deflection of the cantilever versus its position on the sample.

Thermal vibrations of the cantilever are one fundamental source of noise in atomic force microscopy. Thermal noise has already been calculated using the equipartition theorem (see, e.g., [2, 5]). The equipartition theorem states that if a system is in thermal equilibrium every independent quadratic term in its total energy has a mean value equal to $\frac{1}{2}kT$. k is the Boltzmann constant, T is the absolute temperature. When bending the cantilever by a small amount z (in the z direction) its potential energy is $\frac{1}{2}Kz^2$, where K is the spring constant. If the cantilever is supposed to be shaped like a bar of length L with a rectangular cross-section of width w and thickness h (figure 1) its spring constant is [6, 7] $K = 0.25Ewh^3/L^3$. E is the modulus of elasticity. Hence, the equipartition theorem demands $\frac{1}{2}kT = \frac{1}{2}K\widehat{z^2}$. $\widehat{z^2}$ represents the mean square deflection of the cantilever caused by thermal vibrations. Rearrangement yields the amplitude of thermal noise in the vertical direction for a cantilever with a free end:

$$\sqrt{\widehat{z^2}} = \sqrt{\frac{kT}{K}} = \frac{0.64\ \text{Å}}{\sqrt{K}}. \quad (1)$$

The last equation is valid at a temperature of 22 °C with the spring constant in N m^{-1}.

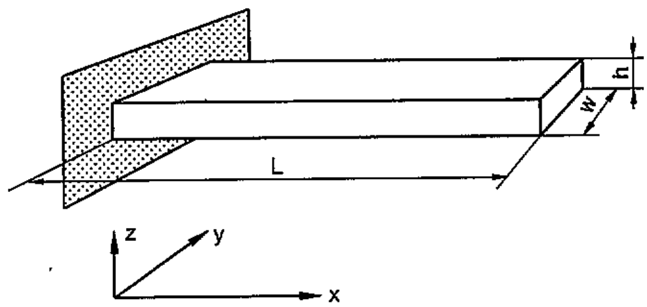

Figure 1. Schematic of the cantilever.

This treatment of thermal noise has one drawback. In many atomic force microscopes cantilever deflection is measured with the optical lever technique. Light of a laser diode is focused onto the back of the cantilever and the reflection angle is measured. With the optical lever technique the inclination at the end of the cantilever $\mathrm{d}z(L)/\mathrm{d}x$ is measured rather than the deflection $z(L)$ itself. In the static situation, that is when the cantilever is deflected by a constant force and the cantilever is not moving, the deflection is related to the inclination by [8]

$$z(L) = \frac{2L}{3}\frac{\mathrm{d}z(L)}{\mathrm{d}x}. \quad (2)$$

For calibration of the instrument a certain deflection $z(L)$ can thus be related to an equivalent change of inclination $\mathrm{d}z(L)/\mathrm{d}x$ by the factor $2L/3$. Hence, when the optical lever technique is applied one is interested in thermal noise of $\mathrm{d}z(L)/\mathrm{d}x$ rather than thermal noise in $z(L)$.

This drawback is especially severe when considering a cantilever which is in contact with a hard surface.

Then thermal noise should be zero since the end of the cantilever, supported by a hard surface, does not move in the z direction. Still, thermal fluctuations cause noise in $\mathrm{d}z(L)/\mathrm{d}x$.

In this paper we calculate thermal noise considering explicitly all possible vibration modes of the cantilever. When taking into account all vibration modes, thermal noise in the deflection $z(L)$ and noise caused by thermal fluctuations in $\mathrm{d}z(L)/\mathrm{d}x$ can be calculated in the same way. We treated two situations. First, the cantilever was supposed to be free at its end. This is the situation when the tip is not in contact with the sample, for instance in non-contact mode, or when the tip images a soft, deformable sample. Second, the end of the cantilever was assumed to be supported. This is the situation when the tip is in contact with a hard, undeformable sample. The real situation lies between these two extremes. Depending on the ratio between spring constant of the cantilever and sample stiffness, the first or the second situation is more realistic.

2. Theory

To obtain thermal noise caused by fluctuations in z and $\mathrm{d}z/\mathrm{d}x$ we use a partial differential equation which describes transversal vibrations of the cantilever. Boundary conditions allow only certain solutions of this differential equation. Each solution, or vibration mode, is characterized by a certain, discrete frequency. In thermal equilibrium each vibration mode has a mean thermal energy of kT ($\frac{1}{2}kT$ for its potential energy plus $\frac{1}{2}kT$ for its kinetic energy). This allows us to calculate the amplitude of thermal noise for each vibration mode. Finally, the total thermal noise was calculated by summing up the contributions of all vibration modes.

Transversal vibrations of a bar in the z direction are described by the partial differential equation [9, 10]

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \frac{Eh^2}{12\rho}\frac{\mathrm{d}^4 z}{\mathrm{d}x^4} = 0. \tag{3}$$

ρ is the density of the cantilever material. Equation (3) neglects damping effects. In addition to equation (3), boundary conditions determine the behaviour of the cantilever. For a bar fixed at one end (at $x = 0$) and free at the other (at $x = L$) the boundary conditions are

$$\Phi(0) = 0 \qquad \frac{\mathrm{d}\Phi(0)}{\mathrm{d}x} = 0$$
$$\frac{\mathrm{d}^2\Phi(L)}{\mathrm{d}x^2} = 0 \qquad \frac{\mathrm{d}^3\Phi(L)}{\mathrm{d}x^3} = 0. \tag{4}$$

The first two conditions are a consequence of fixing the cantilever in amplitude and inclination at $x = 0$. Since the torque of the cantilever is $M(x) \propto \mathrm{d}^2z/\mathrm{d}x^2$ the third boundary condition claims that at $x = L$ the torque vanishes. The fourth condition means that also the external force $F = \mathrm{d}M/\mathrm{d}x \propto \mathrm{d}^3z/\mathrm{d}x^3$ is zero at the end of the cantilever [11].

The solution is

$$z = \sum_{i=1}^{\infty} C_i \sin(\omega_i t + \delta_i)\Phi_i \tag{5}$$

$$\Phi_i = (\sin\alpha_i + \sinh\alpha_i)\left(\cos\frac{\alpha_i}{L}x - \cosh\frac{\alpha_i}{L}x\right) -(\cos\alpha_i + \cosh\alpha_i)\left(\sin\frac{\alpha_i}{L}x - \sinh\frac{\alpha_i}{L}x\right)$$

with

$$\alpha_i^4 = \frac{12\rho\omega_i^2 L^4}{Eh^2}. \tag{6}$$

Each term in the sum represents a vibration mode. It is the product of a time-dependent function and a function which only depends on the position x along the cantilever. C_i is the amplitude of a certain vibration mode. The phase shifts δ_i depend on the initial conditions only. The wavelength of a vibration mode L/α_i is related to the angular vibration frequency ω_i by equation (6).

Only certain discrete values for α_i are allowed. By applying the boundary conditions one finds for the cantilever with a free end that the α_i are determined by $\cos\alpha_i \cosh\alpha_i = -1$. This leads to

$$\alpha_1 = 1.88 \qquad \alpha_2 = 4.69$$
$$\alpha_3 = 7.85 \qquad \alpha_4 = 11.00 \tag{7}$$
$$\alpha_i = (i - \tfrac{1}{2})\pi \qquad \text{for } i \geq 5.$$

For a bar fixed at one end (at $x = 0$) and supported at the other end (at $x = L$) the boundary conditions are

$$\Phi(0) = 0 \qquad \frac{\mathrm{d}\Phi(0)}{\mathrm{d}x} = 0$$
$$\Phi(L) = 0 \qquad \frac{\mathrm{d}^2\Phi(L)}{\mathrm{d}x^2} = 0. \tag{8}$$

In contrast to the free cantilever, the supported cantilever experiences a force at its end. Hence, $F \propto \mathrm{d}^3z/\mathrm{d}x^3$ does not vanish at $x = L$. Therefore $\Phi_i = 0$ at $x = L$. The boundary conditions require that $\sin\alpha_i \cosh\alpha_i = \cos\alpha_i \sinh\alpha_i$ or $\tan\alpha_i = \tanh\alpha_i$. This leads to

$$\alpha_1 = 3.93 \qquad \alpha_2 = 7.07 \qquad \alpha_3 = 10.21$$
$$\alpha_i = (i + \tfrac{1}{4})\pi \qquad \text{for } i \geq 4. \tag{9}$$

Figure 2 shows the first four vibration modes of a cantilever with a free and a supported end.

2.1. Energy of transversal vibrations

To apply the equipartition theorem it is essential to know the vibration energy of the cantilever. Considering only transversal vibrations in the z direction the vibration energy W of a bar is given by [9, 10]

$$W = \frac{Ewh^3}{24}\int_0^L \left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \frac{\rho wh}{2}\int_0^L \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 \mathrm{d}x. \tag{10}$$

The first term accounts for potential, the second for kinetic energy. The integration along the cantilever (in the x

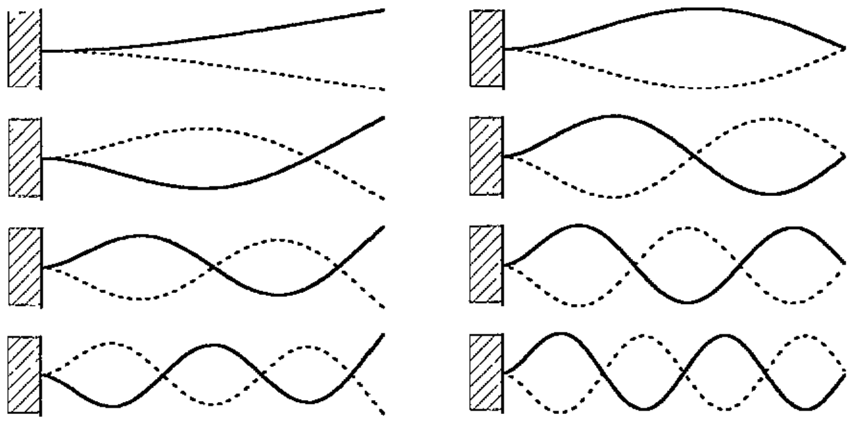

Figure 2. Shape of the first four vibration modes of a free (a) and a supported (b) cantilever.

direction) runs over its entire length L. Inserting the derivatives of the deflection into equation (10) and writing $T_i \equiv \sin(\omega_i t + \delta_i)$ and $T_i^* \equiv \cos(\omega_i t + \delta_i)$ leads to

$$\begin{aligned} W &= \frac{Ewh^3}{24}\int_0^L \left[\sum_{i=1}^{\infty} C_i T_i \frac{\mathrm{d}^2\Phi_i}{\mathrm{d}x^2}\right]^2 \mathrm{d}x \\ &+\frac{\rho wh}{2}\int_0^L \left[\sum_{i=1}^{\infty} C_i \omega_i T_i^* \Phi_i\right]^2 \mathrm{d}x \\ &= \frac{Ewh^3}{24}\Bigg[\sum_{i=1}^{\infty} C_i^2 T_i^2 \int_0^L \left(\frac{\mathrm{d}^2\Phi_i}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \\ &+\sum_{i=1}^{\infty}\sum_{k=1,k\neq i}^{\infty} C_i C_k T_i T_k \int_0^L \frac{\mathrm{d}^2\Phi_i}{\mathrm{d}x^2}\frac{\mathrm{d}^2\Phi_k}{\mathrm{d}x^2}\mathrm{d}x\Bigg] \\ &+\frac{\rho wh}{2}\Bigg[\sum_{i=1}^{\infty} C_i^2 \omega_i^2 T_i^{*2}\int_0^L \Phi_i^2\,\mathrm{d}x \\ &+\sum_{i=1}^{\infty}\sum_{k=1,k\neq i}^{\infty} C_i C_k \omega_i \omega_k T_i^* T_k^* \int_0^L \Phi_i \Phi_k\,\mathrm{d}x\Bigg]. \end{aligned} \qquad (11)$$

In appendix A we further show that

$$\int_0^L \Phi_i \Phi_k\,\mathrm{d}x = \delta_{ik} L I_i \qquad (12)$$

$$\int_0^L \frac{\mathrm{d}^2\Phi_i}{\mathrm{d}x^2}\frac{\mathrm{d}^2\Phi_k}{\mathrm{d}x^2}\,\mathrm{d}x = \delta_{ik}\left(\frac{\alpha_i}{L}\right)^4 L I_i \qquad (13)$$

where $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$ for $i \neq k$. I_i is defined as

$$I_i \equiv \frac{1}{L}\int_0^L \Phi_i^2\,\mathrm{d}x. \qquad (14)$$

In appendix A we show that

$$I_i = (\sin\alpha_i + \sinh\alpha_i)^2. \qquad (15)$$

Applying these relations equation (11) can be simplified:

$$W = \frac{Ewh^3}{24L^3}\sum_{i=1}^{\infty} C_i^2 \alpha_i^4 T_i^2 I_i + \frac{\rho whL}{2}\sum_{i=1}^{\infty} C_i^2 \omega_i^2 T_i^{*2} I_i. \qquad (16)$$

In analogy to the harmonic oscillator we define the variables $q_i \equiv C_i T_i$ and the momenta $p_i \equiv M\,\mathrm{d}q_i/\mathrm{d}t = M\omega_i C_i T_i^*$. With M being the total mass of the cantilever ($M = \rho hwL$) equation (16) further simplifies:

$$W = \frac{K}{2}\sum_{i=1}^{\infty} q_i^2 \alpha_i^4 \frac{I_i}{3} + \frac{1}{2M}\sum_{i=1}^{\infty} p_i^2 I_i. \qquad (17)$$

This equation implies that the vibration energy of the cantilever is the sum of independent quadratic terms in q_i and p_i.

2.2. Mean thermal cantilever deflection for each vibration mode

The deflection at the end of the cantilever is given by $\sum q_i \Phi_i(L)$. Hence, in order to estimate thermal noise it is necessary to know the contribution of each vibration mode. Equation (17) showed that the vibration energy of the cantilever is the sum of independent quadratic terms in q_i and p_i. In thermal equilibrium the equipartition theorem requires that each term has a mean thermal energy of $\frac{1}{2}kT$. Equating $\frac{1}{2}kT$ with $W_i = (K/2)\widehat{q_i^2}\alpha_i^4(I_i/3)$ directly yields

$$\widehat{q_i^2} = \frac{3kT}{\alpha_i^4 K I_i}. \qquad (18)$$

Knowing $\widehat{q_i^2}$, the mean deflection at the end of the cantilever can be calculated. Since

$$\widehat{z_i^2} = \widehat{q_i^2 \Phi_i^2(L)} = \widehat{q_i^2}\Phi_i^2(L) \qquad (19)$$

it is simply

$$\widehat{z_i^2} = \frac{kT}{K}\frac{3\Phi_i^2(L)}{\alpha_i^4 I_i}. \qquad (20)$$

The same result was obtained by Colchero [13]. In appendix B we show that $\Phi_i^2(L)/I_i = 4$. Hence, we finally obtain

$$\widehat{z_i^2} = \frac{12kT}{K\alpha_i^4}. \qquad (21)$$

The amplitude depends only on the spring constant of the cantilever and the temperature. Although each vibration mode has the same mean energy kT the resulting deflection decreases with α_i^{-4}.

When using the optical lever technique the measured thermal noise for a certain vibration mode is determined by the noise in $dz(L)/dx$ multiplied by the calibration factor $2L/3$ (see equation (2)). Since

$$\widehat{z_i^{*2}} = \left(\frac{2L}{3}\right)^2 \widehat{q_i^2 \left(\frac{\Phi_i(L)}{dx}\right)^2} = \left(\frac{2L}{3}\right)^2 \widehat{q_i^2} \left(\frac{\Phi_i(L)}{dx}\right)^2 \tag{22}$$

we simply have

$$\widehat{z_i^{*2}} = \left(\frac{2L}{3}\right)^2 \frac{3kT}{K\alpha_i^4} \frac{(d\Phi_i(L)/dx)^2}{I_i}. \tag{23}$$

The asterisk indicates that z_i^* is a virtual height, caused by the inclination dz/dx and measured with the optical lever technique. In appendix B we show that

$$\frac{(d\Phi_i(L)/dx)^2}{I_i} = 4\frac{\alpha_i^2}{L^2}\left(\frac{\sin\alpha_i \sinh\alpha_i}{\sin\alpha_i + \sinh\alpha_i}\right)^2. \tag{24}$$

Inserting this into equation (23) gives

$$\widehat{z_i^{*2}} = \frac{16kT}{3K\alpha_i^2}\left(\frac{\sin\alpha_i \sinh\alpha_i}{\sin\alpha_i + \sinh\alpha_i}\right)^2. \tag{25}$$

Values of $\sqrt{\widehat{z_i^2}}$ for a cantilever with a free end and of $\sqrt{\widehat{z_i^{*2}}}$ for a free and a supported cantilever at a temperature of 22 °C are given in table 1. $\sqrt{\widehat{z_i^2}}$ decrease more rapidly than values of $\sqrt{\widehat{z_i^{*2}}}$, because $\sqrt{\widehat{z_i^2}}$ is proportional to α_i^{-4} while $\sqrt{\widehat{z_i^{*2}}}$ is only proportional to α_i^{-2}. The amplitude of thermal noise in the deflection $\sqrt{\widehat{z_i^2}}$ falls by a factor of 100 when going from the first to the sixth vibration mode; at the same time the amplitude of the virtual deflection $\sqrt{\widehat{z_i^{*2}}}$ only decreases by a factor of about six.

2.3. Total thermal noise

To calculate the total thermal noise the contributions of all vibration modes have to be considered. The deflection z is the sum of independent terms $q_i\Phi_i$. The probability that the variable q_i assumes a certain value is given by the Boltzmann factor

$$P_i \propto e^{-W_i/kT}.$$

W_i is the potential energy of the vibration mode. W_i is proportional to q_i^2. Hence P_i is a Gauss distribution with regard to q_i. Consequently the mean square deviation of the deflection is the sum of the mean square deviations of each vibration mode [14]:

$$\widehat{z^2} = \sum_{i=1}^{\infty} \widehat{z_i^2} \qquad \widehat{z^{*2}} = \sum_{i=1}^{\infty} \widehat{z_i^{*2}}. \tag{26}$$

For $\widehat{z^2}$ the sum converges rapidly because it is basically a sum over $1/\alpha_i^4$. For $\widehat{z^{*2}}$ the sum converges since for large i

$$\frac{\sin\alpha_i \sinh\alpha_i}{\sin\alpha_i + \sinh\alpha_i} \longrightarrow \frac{\sin\alpha_i\, e^{\alpha_i}/2}{\sin\alpha_i + e^{\alpha_i}/2} \longrightarrow \sin\alpha_i. \tag{27}$$

For the cantilever with a free end $\alpha_i = (i - \frac{1}{2})\pi$ for large i. Hence we have $|\sin\alpha_i| \to 1$. For the supported cantilever $\alpha_i = (i + \frac{1}{4})\pi$ and $|\sin\alpha_i| \to \sin(\pi/4) = 0.707$. In both cases the factor (27) is constant for large i. Hence, the sum $\widehat{z^{*2}} = \sum_{i=1}^{\infty} \widehat{z_i^{*2}}$ is basically a sum over $1/\alpha_i^2$ and converges.

Now we are able to calculate the total thermal noise with equation (26). For the free cantilever thermal fluctuations cause a mean square deflection of

$$\widehat{z^2} = \frac{12kT}{K}\sum_{i=1}^{\infty}\frac{1}{\alpha_i^4}. \tag{28}$$

Rayleigh ([12], p 279) showed that the sum is $\sum_{i=1}^{\infty}(1/\alpha_i^4) = \frac{1}{12}$. Hence, for the free cantilever the total thermal noise is

$$\sqrt{\widehat{z^2}} = \sqrt{\frac{kT}{K}} \tag{29}$$

which agrees with equation (1).

If cantilever deflection is measured with the optical lever technique thermal noise of the virtual deflection is

$$\widehat{z^{*2}} = \frac{16}{3}\frac{kT}{K}\sum_{i=1}^{\infty}\frac{1}{\alpha_i^2}\left(\frac{\sin\alpha_i \sinh\alpha_i}{\sin\alpha_i + \sinh\alpha_i}\right)^2. \tag{30}$$

The sum was calculated numerically. It is

$$\sum_{i=1}^{\infty}\frac{1}{\alpha_i^2}\left(\frac{\sin\alpha_i \sinh\alpha_i}{\sin\alpha_i + \sinh\alpha_i}\right)^2 = \frac{1}{4}$$

for the free cantilever. Hence, we have the final result that when cantilever deflection is measured with the optical lever technique thermal fluctuations of the cantilever cause a virtual mean deflection of

$$\sqrt{\widehat{z^{*2}}} = \sqrt{\frac{4kT}{3K}} = \sqrt{\frac{4}{3}\widehat{z^2}} \tag{31}$$

for a cantilever with a free end. At a temperature of 22 °C this becomes $\sqrt{\widehat{z^{*2}}} = 0.74$ Å$/\sqrt{K}$, where the spring constant has to be inserted in N m^{-1}.

If the end of the cantilever is supported by a hard surface the sum is

$$\sum_{i=1}^{\infty}\frac{1}{\alpha_i^2}\left(\frac{\sin\alpha_i \sinh\alpha_i}{\sin\alpha_i + \sinh\alpha_i}\right)^2 = \frac{1}{16}$$

and thermal noise becomes

$$\sqrt{\widehat{z^{*2}}} = \sqrt{\frac{kT}{3K}}. \tag{32}$$

For the supported cantilever thermal noise is reduced by a factor of two. At 22 °C it is $\sqrt{\widehat{z^{*2}}} = 0.37$ Å$/\sqrt{K}$, with the spring constant in N m^{-1}. Figure 3 shows the dependence of thermal noise on the spring constant.

Table 1. Values for α_i and the contributions of the first six vibration modes to thermal noise. As thermal noise, the square root of the mean square cantilever deflection (or virtual cantilever deflection, if deflection is measured with an optical lever) is given in meters. The values were calculated with equations (1), (31) and (32) for a temperature of 22 °C and a spring constant of 1 N m^{-1}. To obtain the noise contributions for a cantilever with arbitrary spring constant the values have to be divided by $\sqrt{K}$.

i	α_i free	α_i supported	$\sqrt{\widehat{z_i^2}}$ (m) free	$\sqrt{\widehat{z_i^{*2}}}$ (m) free	$\sqrt{\widehat{z_i^{*2}}}$ (m) supported
1	1.8751	3.9266	6.29×10^{-11}	5.77×10^{-11}	2.73×10^{-11}
2	4.6941	7.0686	1.00×10^{-11}	3.20×10^{-11}	1.47×10^{-11}
3	7.8548	10.2102	3.58×10^{-12}	1.88×10^{-11}	1.02×10^{-11}
4	10.9955	13.3518	1.83×10^{-12}	1.34×10^{-11}	7.81×10^{-12}
5	14.1372	16.4934	1.11×10^{-12}	1.04×10^{-11}	6.32×10^{-12}
6	17.2788	19.6350	7.41×10^{-13}	8.53×10^{-12}	5.31×10^{-12}

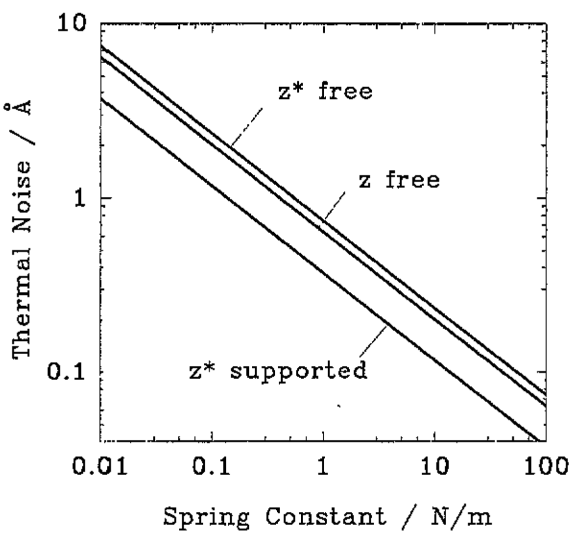

Figure 3. Total thermal noise (square root of the mean square cantilever deflection) versus the spring constant. The curves show thermal noise of the deflection $z(L)$ for a free cantilever (calculated with equation (1)) and noise in the virtual deflection $z^*(L)$ caused by thermal fluctuations in $dz(L)/dx$ for a free (equation (31)) and a supported cantilever (equation (32)). All calculations were done for a temperature of 22 °C.

3. Discussion

Equation (3) is only valid under several assumptions. First the wavelength of a vibration mode needs to be longer than the thickness of the cantilever. With a typical length of 100 μm and a thickness of 1 μm calculations are reasonable up to the 30th vibration mode. Equation (3) also assumes that the inclination of the cantilever is much smaller than unity. With typical noise amplitudes in the order of 1 Å and for a wavelength of greater than 1 μm this assumption is always fullfilled.

The amplitude of thermal deflection decreases with increasing vibration numbers. This is the reason why the sum of the contributions of thermal deflection in equations (28) and (30) converges. The first 10 vibration modes already contribute 99.93% to $\widehat{z^2}$ and 96% to $\widehat{z^{*2}}$ for the free cantilever. Hence, even if for higher vibration mode equation (3) becomes invalid the limits in equations (28) and (30) are good approximations for the amplitudes of thermal noise.

An important limitation is the shape of the cantilever, which was supposed to be a bar with rectangular cross-section. Often cantilevers are 'V' shaped with a triangular end. To our knowledge no calculations of the dynamic behaviour of 'V' shaped cantilevers have been done. Spring constants of 'V' shaped cantilevers are normally calculated by adding the spring constants of both arms [15, 16]. This might lead to errors of 25% [17]. In addition, the relation (2) between deflection and inclination of the cantilever depends on the shape of the cantilever. This introduces another error of typically 10% [8].

Hutter and Bechhoefer [18] suggested measuring the amplitude of thermal noise in order to obtain the spring constant. The authors always applied equation (1) to calculate the spring constant. However, if cantilever deflection is measured with an optical lever equation (31) has to be applied instead.

Acknowledgments

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Appendix A.

For the following it is important to note that each vibration mode fulfills the differential equation

$$\frac{d^4\Phi}{dx^4} = \left(\frac{\alpha}{L}\right)^4 \Phi \qquad \text{(A1)}$$

which can easily be verified by inserting $C_i \Phi_i \sin(\omega_i t + \delta_i)$ into equation (3).

Rayleigh ([12], p 263) has already shown that all mixed integrals $\int_0^L \Phi_i \Phi_k \, dx$ are zero. In analogy it can be shown that also the mixed integrals

$$\int_0^L \frac{d^2\Phi_i}{dx^2}\frac{d^2\Phi_k}{dx^2}dx$$

vanish. All functions $\Psi_i(x) = d^2\Phi_i/dx^2$ are solutions of equation (A1), which can easily be shown by insertion. To show that the functions Ψ_i are orthogonal one can calculate the integral

$$\begin{aligned}\frac{\alpha_i^4-\alpha_k^4}{L^4}\int_0^L \Psi_i\Psi_k dx &= \int_0^L \left(\frac{d^4\Psi_i}{dx^4}\Psi_k - \frac{d^4\Psi_k}{dx^4}\Psi_i\right)dx\\
&= \left[\Psi_k\frac{d^3\Psi_i}{dx^3} - \Psi_i\frac{d^3\Psi_k}{dx^3} + \frac{d\Psi_i}{dx}\frac{d^2\Psi_k}{dx^2} - \frac{d\Psi_k}{dx}\frac{d^2\Psi_i}{dx^2}\right]_0^L\\
&= \left[\frac{d^2\Phi_k}{dx^2}\frac{d^5\Phi_i}{dx^5} - \frac{d^2\Phi_i}{dx^2}\frac{d^5\Phi_k}{dx^5}\right.\\
&\quad \left.+\frac{d^3\Phi_i}{dx^3}\frac{d^4\Phi_k}{dx^4} - \frac{d^3\Phi_k}{dx^3}\frac{d^4\Phi_i}{dx^4}\right]_0^L\\
&= \left[\frac{\alpha_i^4}{L^4}\frac{d^2\Phi_k}{dx^2}\frac{d\Phi_i}{dx} - \frac{\alpha_k^4}{L^4}\frac{d^2\Phi_i}{dx^2}\frac{d\Phi_k}{dx}\right.\\
&\quad \left.+\frac{\alpha_k^4}{L^4}\frac{d^3\Phi_i}{dx^3}\Phi_k - \frac{\alpha_i^4}{L^4}\frac{d^3\Phi_k}{dx^3}\Phi_i\right]_0^L.\end{aligned} \quad (A2)$$

Inserting the boundary conditions it can be seen that the integral is indeed zero for $i \neq k$.

Now we show that

$$\int_0^L \left(\frac{d^2\Phi_i}{dx^2}\right)^2 dx = \frac{\alpha_i^4}{L^4}\int_0^L \Phi_i^2 dx. \quad (A3)$$

Therefore the first integral is twice partially integrated:

$$\begin{aligned}\int_0^L \left(\frac{d^2\Phi_i}{dx^2}\right)^2 dx &= \left[\frac{d\Phi_i}{dx}\frac{d^2\Phi_i}{dx^2} - \Phi_i\frac{d^3\Phi_i}{dx^3}\right]_0^L\\
&\quad + \int_0^L \Phi_i\frac{d^4\Phi_i}{dx^4}dx.\end{aligned} \quad (A4)$$

Using equation (A1) we obtain

$$\begin{aligned}\int_0^L \left(\frac{d^2\Phi_i}{dx^2}\right)^2 dx &= \left[\frac{d\Phi_i}{dx}\frac{d^2\Phi_i}{dx^2} - \Phi_i\frac{d^3\Phi_i}{dx^3}\right]_0^L\\
&\quad + \frac{\alpha_i^4}{L^4}\int_0^L \Phi_i^2 dx.\end{aligned} \quad (A5)$$

Inserting the boundary conditions (4) or (8) shows that the term in square brackets on the right is zero.

To calculate the integral I_i we substitute the original integration variable x by $\xi \equiv x/L$. Then

$$\begin{aligned}I_i = \int_0^1 &[(\sin\alpha_i + \sinh\alpha_i)(\cos\alpha_i\xi - \cosh\alpha_i\xi)\\
&-(\cos\alpha_i + \cosh\alpha_i)(\sin\alpha_i\xi - \sinh\alpha_i\xi)]^2 dx.\end{aligned} \quad (A6)$$

Multiplying out and integrating each of the resulting 10 terms separately yields

$$\begin{aligned}\alpha_i I_i &= (\sin\alpha_i + \sinh\alpha_i)^2\\
&\quad\times\left(\alpha_i + \frac{\sin 2\alpha_i}{4} + \frac{\sinh\alpha_i\cosh\alpha_i}{2}\right.\\
&\quad\left. + \cos\alpha_i\sinh\alpha_i + \sin\alpha_i\cosh\alpha_i\right)\\
&\quad -(\sin\alpha_i\cos\alpha_i + \sinh\alpha_i\cosh\alpha_i\\
&\quad + \sin\alpha_i\cosh\alpha_i + \sinh\alpha_i\cos\alpha_i)\\
&\quad\times\left(\sin^2\alpha_i + \frac{\cosh^2\alpha_i}{2} + 2\sin\alpha_i\sinh\alpha_i - \frac{1}{2}\right)\\
&\quad + (\cos\alpha_i + \cosh\alpha_i)^2\\
&\quad\times\left(-\frac{\sin 2\alpha_i}{4} + \frac{\sinh\alpha_i\cosh\alpha_i}{2}\right.\\
&\quad\left. - \cos\alpha_i\sinh\alpha_i + \sin\alpha_i\cosh\alpha_i\right).\end{aligned} \quad (A7)$$

With $\cosh^2\alpha_i = \sinh^2\alpha_i + 1$ the second factor in the second term is equal to

$$\sin^2\alpha_i + 2\sin\alpha_i\sinh\alpha_i + \sinh^2\alpha_i = (\sin\alpha_i + \sinh\alpha_i)^2.$$

Using this several terms cancel out and expression (A7) becomes:

$$\begin{aligned}\alpha_i I_i &= (\sin\alpha_i + \sinh\alpha_i)^2\\
&\quad\times\left(\alpha_i - \frac{\sinh\alpha_i\cosh\alpha_i}{2} - \frac{\sin\alpha_i\cos\alpha_i}{2}\right)\\
&\quad + (\cos\alpha_i + \cosh\alpha_i)^2\\
&\quad\times\left(\frac{\sinh\alpha_i\cosh\alpha_i}{2} - \frac{\sin\alpha_i\cos\alpha_i}{2}\right.\\
&\quad\left. - \cos\alpha_i\sinh\alpha_i + \sin\alpha_i\cosh\alpha_i\right).\end{aligned} \quad (A8)$$

Multiplying out and substituting $1 - \cos^2\alpha_i$ for $\sin^2\alpha_i$ and $\cosh^2\alpha_i - 1$ for $\sinh^2\alpha_i$ equation (A8) simplifies:

$$\begin{aligned}\alpha_i I_i &= \alpha_i(\sin\alpha_i + \sinh\alpha_i)^2 - \cos\alpha_i\sinh\alpha_i\\
&\quad + \sin\alpha_i\cosh\alpha_i + \sin\alpha_i\cos\alpha_i\cosh^2\alpha_i\\
&\quad - \cos^2\alpha_i\sinh\alpha_i\cosh\alpha_i.\end{aligned} \quad (A9)$$

Now we use the boundary conditions. For a cantilever with a free end the boundary conditions require that $\cos\alpha_i\cosh\alpha_i = -1$. Inserting this yields

$$\begin{aligned}\alpha_i I_i &= \alpha_i(\sin\alpha_i + \sinh\alpha_i)^2\\
&\quad - \cos\alpha_i\sinh\alpha_i + \sin\alpha_i\cosh\alpha_i\\
&\quad - \sin\alpha_i\cosh\alpha_i + \cos\alpha_i\sinh\alpha_i\\
&= \alpha_i(\sin\alpha_i + \sinh\alpha_i)^2.\end{aligned} \quad (A10)$$

For a cantilever with a supported end the boundary conditions require that $\cos\alpha_i\sinh\alpha_i = \sin\alpha_i\cosh\alpha_i$. Inserting this into the third and fifth terms of equation (A9) yields

$$\begin{aligned}\alpha_i I_i &= \alpha_i(\sin\alpha_i + \sinh\alpha_i)^2\\
&\quad - \cos\alpha_i\sinh\alpha_i + \cos\alpha_i\sinh\alpha_i\\
&\quad + \sin\alpha_i\cos\alpha_i\cosh^2\alpha_i - \sin\alpha_i\cos\alpha_i\cosh^2\alpha_i\\
&= \alpha_i(\sin\alpha_i + \sinh\alpha_i)^2.\end{aligned} \quad (A11)$$

Hence, in both cases we obtain the result

$$I_i = (\sin\alpha_i + \sinh\alpha_i)^2. \quad (A12)$$

Appendix B. Calculation of $\Phi_i^2(L)/I_i$ and $(1/I_i)\left(\mathrm{d}\Phi_i(L)/\mathrm{d}x\right)^2$

Thermal noise of the free cantilever is determined by the factor $\Phi_i^2(L)/I_i$. Therefore one needs to know $\Phi_i^2(L)$. Inserting L into the expression for Φ_i of equation (5) and multiplying out leads to

$$\Phi_i(L) = 2\cos\alpha_i \sinh\alpha_i - 2\sin\alpha_i \cosh\alpha_i$$

and $\Phi_i^2(L)$ becomes

$$\begin{aligned}\frac{\Phi_i^2(L)}{4} &= \cos^2\alpha_i \sinh^2\alpha_i \\ &\quad -2\cos\alpha_i \sin\alpha_i \cosh\alpha_i \sinh\alpha_i + \sin^2\alpha_i \cosh^2\alpha_i \\ &= \sinh^2\alpha_i - \sin^2\alpha_i \sinh^2\alpha_i + \sin^2\alpha_i \sinh^2\alpha_i \\ &\quad + \sin^2\alpha_i - 2\cos\alpha_i \sin\alpha_i \cosh\alpha_i \sinh\alpha_i \\ &= \sinh^2\alpha_i + \sin^2\alpha_i - 2\cos\alpha_i \sin\alpha_i \cosh\alpha_i \sinh\alpha_i .\end{aligned} \tag{B1}$$

Using $\cos\alpha_i \cosh\alpha_i = -1$, the requirement for a free cantilever, this becomes

$$\frac{\Phi_i^2(L)}{4} = (\sin\alpha_i + \sinh\alpha_i)^2.$$

Since $(\sin\alpha_i + \sinh\alpha_i)^2 = I_i$ the ratio $\Phi_i^2(L)/I_i$ equals four.

When the optical lever technique is used to measure the deflection of the cantilever, the factor $(\mathrm{d}\Phi_i(L)/\mathrm{d}x)^2/I_i$ is relevant. Since

$$\begin{aligned}\frac{L}{\alpha_i}\frac{\mathrm{d}\Phi_i(L)}{\mathrm{d}x} &= (\sin\alpha_i + \sinh\alpha_i)(-\sin\alpha_i - \sinh\alpha_i) \\ &\quad -(\cos\alpha_i + \cosh\alpha_i)(\cos\alpha_i - \cosh\alpha_i) \\ &= -\sin^2\alpha_i - 2\sin\alpha_i \sinh\alpha_i - \sinh^2\alpha_i \\ &\quad -\cos^2\alpha_i + \cosh^2\alpha_i \\ &= -2\sin\alpha_i \sinh\alpha_i\end{aligned} \tag{B2}$$

this factor becomes

$$\frac{(\mathrm{d}\Phi_i(L)/\mathrm{d}x)^2}{I_i} = 4\frac{\alpha_i^2}{L^2}\left(\frac{\sin\alpha_i \sinh\alpha_i}{\sin\alpha_i + \sinh\alpha_i}\right)^2. \tag{B3}$$

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