Sigmas, Deltas and Epsilons

Suggested reading:
Weber and Arfken, Chapter 2, Section 2.9, pages 153 - 157.

There are a few notational conventions which make some vector calculations much easier to do. These include the “Kronecker delta”, \( \delta_{ij} \) (which should not be confused with the “Dirac delta function” \( \delta(x) \) which we will introduce later) and the Levi-Civita symbol \( \epsilon_{ijk} \).

The Kronecker Delta: \( \delta_{ij} \)

When dealing with orthonormal basis vectors in Linear Vector Spaces if we take the scalar product of any two of these basis vectors \( \hat{e}_p \) and \( \hat{e}_q \) we can only get a 0 or a 1 depending on whether \( p = q \) or \( p \neq q \). Thus, we define the Kronecker delta symbol

\[
\delta_{pq} \equiv \hat{e}_p \cdot \hat{e}_q = \begin{cases} 
0 & \text{if } p \neq q, \\
1 & \text{if } p = q.
\end{cases}
\]  

we will make extensive use of this symbol in this course (and you will see it in many other physics courses as well).
The Levi-Civita Symbol: $\epsilon_{ijk}$

In evaluation vector products (and in proving vector identities) it is very useful to use the Levi-Civita “completely anti-symmetric tensor” defined by

$$
\epsilon_{pqr} \equiv \begin{cases} 
0 & \text{if any two (or more) indices are equal,} \\
+1 & \text{if (p,q,r) is an even permutation of (1,2,3),} \\
-1 & \text{if (p,q,r) is an odd permutation of (1,2,3).}
\end{cases}
$$

(2)

for example, $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$ and $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$.

Using the Levi-Civita symbol, the vector components $C_p$ of the cross product of two vectors $\vec{C} = \vec{A} \times \vec{B}$

$$
C_p = \sum_{q,r} \epsilon_{pqr} A_q B_r
$$

Explicitly, for any p, say $p = x$,

$$
C_x = \sum_{q,r} \epsilon_{xqr} A_q B_r = \epsilon_{xyz} A_y B_z + \epsilon_{xzy} A_z B_y
$$

(3)

all other terms being zero because of the definition of $\epsilon_{pqr}$. 

$$
= A_y B_z - A_z B_y
$$
An illustration of how this symbol is used will convince you of its utility.

**Example: Prove that** \( \vec{\nabla} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{\nabla} \times \vec{u}) - \vec{u} \cdot (\vec{\nabla} \times \vec{v}) \)

Using the Levi-Civita notation, the “i’th” component of the cross product of the two vectors is

\[
(\vec{u} \times \vec{v})_i = \sum_{jk} \epsilon_{ijk} u_j v_k
\]

Taking the scalar product with \( \vec{\nabla} \) gives

\[
\vec{\nabla} \cdot (\vec{u} \times \vec{v}) = \sum_i \sum_{jk} \epsilon_{ijk} \frac{\partial}{\partial x_i} (u_j v_k)
\]

\[
= \sum_i \sum_{jk} \epsilon_{ijk} [\left( \frac{\partial u_j}{\partial x_i} \right) v_k + u_j \left( \frac{\partial v_k}{\partial x_i} \right)]
\]

\[
= \sum_k v_k \left( \sum_{ij} \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} \right) + \sum_j u_j \left( \sum_{ik} \epsilon_{ijk} \frac{\partial v_k}{\partial x_i} \right)
\]

\[
= \sum_k v_k \left( \sum_{ij} \epsilon_{kij} \frac{\partial u_j}{\partial x_i} \right) - \sum_j u_j \left( \sum_{ik} \epsilon_{jik} \frac{\partial v_k}{\partial x_i} \right)
\]

\[
= \vec{v} \cdot (\vec{\nabla} \times \vec{u}) - \vec{u} \cdot (\vec{\nabla} \times \vec{v})
\]

Here we have used the fact that the scalar product of two vectors is \( \vec{a} \cdot \vec{b} = \sum_k a_k b_k \). The most confusing line in this proof is undoubtedly the second to last line. Here, as is highlighted in blue and red colours, we have made the substitution \( \epsilon_{kij} = \epsilon_{ijk} \) and \( \epsilon_{jik} = -\epsilon_{ijk} \). Having done that, we need simply look at the definition of the
cross product of two vectors using the Levi-Civita notation to see that the equation hold as shown. One other point that you need to get used to is the idea that the “summation indices”, such is i,j,k or p,q,r are “dummy indices”. This means that I can change from “i” to “p” or “ℓ”, etc, any time I wish. The only thing is not to become confused!

In more complex problems it is useful to know that

\[ \sum_k \epsilon_{ijk} \epsilon_{\ell mk} = \delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell} \]

You should try to prove this statement (it isn’t difficult, just consider all possible cases remembering that the indices \( i, j, k; \ell, m, k \) can only take on the values 1, 2, 3.